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FOUNDATIONS OF THE LOGICAL THEORY OF SCIENTIFIC KNOWLEDGE (COMPLEX LOGIC)

A. A. ZINOV'EV

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VOLUME IX

Edited by Robert S. Cohen and Marx W. Wartofsky

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SOCIAL AND BEHAVIORAL SCIENCES

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FOUNDATIONS OF THE LOGICAL THEORY OF SCIENTIFIC KNOWLEDGE (COMPLEX LOGIC)

Revised and Enlarged English Edition

with an Appendix by

G. A. Smirnov, E. A. Sidorenko, A. M. Fedina, and L. A. Bobrova

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EDITORIAL INTRODUCTION

Boston Studies in the Philosophy of Science are devoted to symposia, congresses, colloquia, monographs and collected papers on the philosophical foundations of the sciences. It is now our pleasure to include A. A. Zinov'ev's treatise on complex logic among these volumes. Zinov'ev is one of the most creative of modern Soviet logicians, and at the same time an innovative worker on the methodological foundations of science. Moreover, Zinov'ev, although still a developing scholar, has exerted a substantial and stimulating influence upon his colleagues and students in Moscow and within other philosophical and logical circles of the Soviet Union. Hence it may be helpful, in bringing this present work to an English-reading audience, to review briefly some contemporary Soviet investigations into scientific methodology.

During the 1950's, a vigorous new research program in logic was undertaken, and the initial published work – characteristic of most Soviet publications in the logic and methodology of the sciences – was a collection of essays, *Logical Investigations* (Moscow, 1959). Among the authors, in addition to Zinov'ev himself, were the philosophers A. Kol'man and P. V. Tavanec, and the mathematicians and linguists, S. A. Janovskaja, A. S. Esenin-Vol'pin, S. K. Šaumjan, G. N. Povarov. Two principal themes dominate this work: first, that the *results* of mathematical logic have practical importance for the sciences (and that modern logic may be understood as of general philosophical significance); and second, that it is impossible to provide a successful and profound methodology of science without using the formal apparatus and methods of contemporary logic. This relationship, between research into mathematical and formal logic on the one hand and the methodology of the sciences on the other, became a principal focus of logical research in the Soviet Union during the subsequent period, and it remains so at present.

Perhaps the pre-eminent group of Soviet logicians and methodologists is gathered together in Moscow at the Institute of Philosophy of the Academy of Sciences of the U.S.S.R. Among its number are P. V. Tavanec, D. P. Gorskij, G. I. Ruzavin, V. A. Smirnov, A. A. Zinov'ev. These scholars, together with other colleagues in the section on logic of the Institute of Philosophy have been the most productive of Soviet logicians and methodologists; they have published a series of volumes which, in effect, has fulfilled the function of a periodical of logical and methodological studies.

Among the most interesting of these volumes, in addition to the *Logical Investigations* of 1959 have been: *Philosophical Questions of Contemporary Formal Logic*, 1962; *Problems of the Logic of Scientific Knowledge*, 1964 (English translation, 1970, D. Reidel); *Logical Structure of Scientific Knowledge*, 1966; *Logical Semantics and Modal Logics*, 1967; *Investigation of Logical Systems*, 1970; *Non-Classical Logics*, 1970.

Further, a number of individual monographs have been published, including: N. I. Stjažkin, *History of Mathematical Logic from Leibniz to Peano*, 1964 (English translation, 1969, MIT Press); A. I. Uemov, *Things, Properties and Relations*, 1963; D. P. Gorskij, *Questions of Abstraction and Notions of Construction*, 1961; and a number of monographs by Zinov'ev himself (see below).

Logical research has also been undertaken in the sections on logic of the departments of philosophy of the Moscow State University and the University of Leningrad by E. K. Vojšvillo, A. A. Ivin, O. F. Serebrjannikov among others, and also at other Soviet academic institutions: for example, A. I. Uemov at the University of Odessa, V. N. Sadovskij at the Institute of the History and Theory of the Natural Sciences and Technology of the Soviet Academy in Moscow, M. V. Popovič at the Institute of Philosophy of the Ukrainian Academy of Sciences in Kiev, and many others. The literature has been voluminous but two works from Zinov'ev's own colleagues particularly should be noted: O. F. Serebrjannikov, *Heuristic Principles and Logical Calculi*, 1970, and A. A. Ivin, *Foundations of the Logic of Value*, 1970.

Despite the immense effort devoted to the elaboration of the technical apparatus of formal logic, particularly by Zinov'ev, Smirnov, and Serebrjannikov, the primary aim of this trend in Soviet logic has been the application of formal logic to the decisive solution of a range of problems in the methodology of the sciences. Among Soviet scholars, this topic is called the 'logic of science'. Zinov'ev's group of colleagues has been formed during the past few years. It consists of former research students of

Zinov'ev who either continue to collaborate with him or to work independently on the further development of his ideas. They include A. A. Ivin at the Department of Philosophy of the Moscow State University, G. A. Smirnov at the Institute of the History and Theory of the Natural Sciences in Moscow, H. Vessel of the Department of Philosophy at the Humboldt University in Berlin, E. A. Sidorenko and A. M. Fedina at the Institute of Philosophy in Moscow, L. A. Bobrova, Dept. of Logic, Moscow State University.

The distinctive nature of the logical investigations of Zinov'ev and his colleagues are quite fully expounded in the present book. It is, briefly, the attempt to construct a particular logical conception of contemporary logic, broad enough in scope to encompass the whole range of issues beginning with a general theory of signs and concluding with logical analyses of such scientific problems as motion, causality, space and time.

In Zinov'ev's conception, the theory of deduction has a central place. But this theory is carefully to be distinguished from generally accepted theoretical interpretations in the systems of classical and intuitionist mathematical logic. As he writes in an earlier essay, which may be taken as a technical introduction to this monograph ('Logical and Physical Implication', p. 91);

> Under the influence of the mathematization of sciences and the successes of mathematical logic in the last few decades, a special branch of logical-philosophical research has developed. Its essence is the use of the ideas, the apparatus (calculi) and methods of mathematical logic and mathematics (exact methods) in the solution of a series of traditional problems of formal logic and philosophy as well as of new problems of the methodology of science specifically connected with the development of contemporary science.
>
> In this branch one considers the epistemological interpretation of formal systems of logic, constructs formal systems for the express purpose of describing various aspects of human cognitive activity, solves certain problems of philosophy by means of logical-mathematical constructions, and uses the accomplishments of logic to overcome philosophical difficulties in the natural sciences.
>
> Among the problems thus researched we find causal and nomological statements, scientific laws, operational and inductive definitions, models, reasons for and means of limiting the rules of judgment in various domains of science, construction and interrelating of theory, etc. There are numerous works on these subjects, the study of each of which requires definite specialisation.
>
> There are philosophers, logicians, and mathematicians who, for a variety of reasons, are inclined to exclude this branch of logical-philosophical investigation from the sphere of philosophy. On the other hand, there are others who hold that the application of exact methods in philosophy does not fall outside the realm of philosophy if the results thereof are strictly compared with the previous methods and results of philoso-

phy. Regardless of the outcome of this dispute, the fact remains that exact methods are applied to problems which have always been considered philosophical.

Efforts to use exact methods in philosophy proper are found in the works of the positivist philosophers who reduced the problems of philosophy to problems of formal logic. Whence the impression that the application of such methods in philosophy is a mark of positivism. This view is incorrect. The use of exact methods in philosophy (as the refusal to use them) in itself says nothing [about] the philosophical position of the user. And if there are erroneous philosophical views attached to such use, they can be successfully opposed not by ignoring exact methods in solving philosophical problems but rather by carefully and expertly using them and by developing new methods of this type.

The use of exact methods as a possible mode of philosophical investigation can, if the object and tasks of philosophy are properly understood, lead to great progress in bringing it into consonance with the thought-structure of contemporary science. The application of these methods marks a transition to the theoretical level in the solution of philosophical problems. At this level, new knowledge [of] the objects of investigation comes not through observation and experiment (as happens on the empirical level) but through logical judgments in the framework of a given or newly developed theory (i.e., special groups of concepts and statements united by rules of logic). The value of the theoretical level is well known and we need not discuss it here. The same is true in philosophy (not as an object about which one can talk but as a means of investigation).

As regards the question [of] the non-reducibility of philosophy to formal logic, the application of these methods makes it possible not just to declare this as a preconceived notion but strictly to demonstrate it for any philosophical problem.

and a few pages later,

In the wide sense of the term the problem of logical implication can be formulated as follows: is a given logical construction suited to the description of the properties of logical implication? Do the formulae of a given formal construction of logic correspond to the intuitive understanding of logical implication? By intuitive understanding of logical implication we here mean the understanding which grows up in people perforce of habitually judging (reasoning, drawing conclusions) and observing such activity in others. The habit of judging according to the rules of logical implication comes as the result of personal experience, education and acquired science. What form must a logical system have in order to satisfy the intuitive understanding of logical implication?

One here talks about intuitive understanding because logic presents the results of its investigations in the form of an apparatus for the practical use of those who judge, infer, reason, prove, etc. And they, of course, compare these results precisely with their usually clear understanding of the rules of these operations. Of course, intuitive understanding is not something once and for all given and absolutely universal. But there are, all the same, some stable and general aspects and they suffice for mutual understanding. What is more, the task of logical constructions is not limited to following intuition passively. Their basic task consists in clarifying and standardizing intuitive understanding, in systematizing the rules of logical implication, in providing the means of establishing their reliability and the means of predicting such reliable rules which have not yet been met in the experience of judgment or which have not been actively realized. What form must a logical system have so that it – without thereby being limited – will be as close as possible to the intuitive bases?

After the appearance in 1967 of the first edition of the present work, Zinov'ev published several further studies. Based upon discussion (by R.S.C.) with him during Summer 1971, we can briefy mention these new results. The most recent, *The Logic of Science*, 1971, mainly coincides with the present work. However, in it the author provides a considerably more detailed and thereby more easily grasped explanation of the basic principles and fundamental notions of his conception of logic; moreover he expounds this conception in the 1971 work with a minimal formal apparatus. It should be accessible to a much wider circle of readers. In addition, the new book has an extensive section devoted to the methodology of physical science with elaborate studies of space, time, causality, motion, etc. There, Zinov'ev criticizes the point of view which claims that a special logic [quantum logic] is necessary for micro-physics, different from the logical and methodological formalism of macro-physics. In a related section, Zinov'ev expounds his conception of the universality of logic, by which he understands the independence of logical rules from the specific domains of application of these rules in the sphere of objects.

In the 1971 book, Zinov'ev proceeds from an analysis of 'ontological terminology' (his phrase), perhaps more easily identified as 'physical terminology', and from his exposition of logical rules of operation with these ontological terms, to his major conclusions. Many problems which are discussed in the philosophy of physics, and which are particularly connected with modern discoveries, are shown to be only terminological, independent of the success or inadequacy of physics proper. Such, for example, is the central problem of the reversibility of time. Indeed, in Zinov'ev's analysis, many assertions which traditionally have been construed to be empirical or physical, turn out to be the implicit consequences of definitions of terms; or at any rate they may be conceived thus without contradiction or empirical refutation. An example is the assertion that a body cannot be in different places at the same time.

In another work, his newer *Complex Logic* published in 1970, Zinov'ev offers a systematic account of the formal apparatus of logical implication. Here the most interesting part is perhaps his theory of quantifiers. Zinov'ev formulates the entire range, the totality, of different logical systems of the theory of quantifiers which satisfy differing corresponding intuitive premises, and he investigates their properties. In particular, the 1970 monograph provides a fuller investigation of the strict theory of quantifiers.

Evidently, Zinov'ev's conception of logic is opposed to certain intellectual trends of contemporary logic and methodology of science. He ruefully contrasts the insignificance of many problems treated in the standard methodology of science with the grandiosity of methodological claims, and with the tendency to apply methodological formulations beyond their domain. And he is critical of the misuse of deductive logic and the accepted formulations of the methodology of natural science in social-scientific investigations.

A. A. Zinov'ev foresees fruitful applications of his conception of 'complex logic' throughout methodological investigations into the natural and the social sciences, and in the theory of values as well, but he also sees the need for extensive further work in pure logic. We warmly anticipate his ongoing investigations.

Boston University Center for the
Philosophy and History of Science
Spring 1972

R. S. COHEN
M. W. WARTOFSKY

NOTE

Some of Zinov'ev's studies have appeared in English:

[*SSP*: *Soviet Studies in Philosophy* (International Arts and Sciences Press, White Plains, New York)]

1. *Philosophical Problems of Many-Valued Logic* (ed. and transl. by G. Kung and D. D. Comey), Humanities Press, New York; and D. Reidel, Dordrecht, Holland; 1963.

2. 'Logical and Physical Implication', in: *Problems of the Logic of Scientific Knowledge*, (ed. by P. V. Tavanec), pp. 91–159 (transl. by T. J. Blakeley) (Humanities Press, New York; and D. Reidel, Dordrecht, Holland; 1970). Original: *Problemy logiki naučnogo poznanija*, Moscow, 1964.

3. 'Two-Valued and Many-Valued Logic', *SSP* 2, 69–84 (Summer-Fall 1963); from: *Filosofskie voprosy sovremennoj formal'noj logiki* (ed. by P. V. Tavanec), Institute of Philosophy, U.S.S.R. Academy of Sciences, Moscow, 1962.
(Note: 7 of the 12 papers in the original volume appeared in this issue of *SSP*.)

4. 'On the Application of Modal Logic in the Methodology of Science', *SSP* 3, 20–26 (Winter 1964–65); from *Voprosy filosofii*, 1964, No. 8.

5. 'On Classical and Non-Classical Situations in Science', *SSP* 7, 24–33 (Spring 1969); from: *Voprosy filosofii*, 1968, No. 9.

6. 'On the Logic of Microphysics', *SSP* **9**, 222–236 (Winter 1970–71); from: *Voprosy filosofii* 1970, No. 2 (Part of a symposium on Logic and Quantum Mechanics, with other contributions by B. G. Kuznecov: 'On Quantum-Relativistic Logic'; R. A. Aronov: 'Toward a Logic of the Microworld'; I. P. Staxanov: 'The Logic of "Possibility"').

PREFACE

Logical theory of scientific knowledge is the investigation of scientific knowledge within the framework of the concepts and methods of logic. The bases for such investigations in contemporary logic were provided by Frege, Russell, Lewis, Łukasiewicz, Carnap, Reichenbach, Tarski, Ajdukiewicz, and many other scientists, whose works are generally quoted in logical-philosophical writings.

The present work offers a somewhat systematic construction of that conception of the logical theory of scientific knowledge which was to be found in incomplete form in the author's earlier works. This construction has to do only with the fundamentals of the theory of scientific knowledge. Therefore, the book is to be taken neither as a textbook nor as a presentation of what is generally done in corresponding branches of logic.

Some details of this way of looking at things are truisms to be found in most other works on the same subject. But in its general character and on the most essential points it is essentially different from such other works, as the reader can see by carrying out the comparison.

The basic object of this book is to present as simply and systematically as possible the ideas and principles which seem to us to be the most promising for the theory of scientific knowledge. Therefore, the formal logical apparatus which could be developed on this basis has been held to an absolute minimum.

Mathematical logic has carried the day in the theory of scientific knowledge. But one finds in logical circles a prejudice that mathematical logic as it is found in textbooks (propositional calculus and predicate calculus, with some expansions) is the only possible logical apparatus for the solution of all problems of the theory of scientific knowledge. The fact of the matter is that mathematical logic in its normal form is only a fragment of the theory of scientific knowledge and the other sections cannot be reduced to it. This is particularly true of the theory of terms, of the forms of logical entailment, syllogistics, physical entailment, and other sections of logic which are stressed in the present volume.

This could be called "complex logic" for the following reasons. Contemporary logic has developed into an extensive and sophisticated science. It has need of systematization. This is not simply a question of a suitable presentation of its results in teaching. It is more an effort to find a notion of logic itself such that the various calculi, theories, trends, etc., appear as natural fragments of a single system. Our present effort is in this direction. In particular, classical logic, intuitionist logic, the system of strong implication, and other logical systems, which are usually taken as different solutions of one and the same problem of the definition of the rules of logical inference, are here viewed as solutions of different problems, i.e., as different fragments of a single logical system. For this purpose we need a unified logical structure – a type of logical "base" – which must contain the various logical calculi and which must itself have the form of a deductive system. Complex logic is intended for this purpose. Further, the method of construction we have chosen – i.e., the construction of the different branches of logic through appropriate additions to the general theory of deduction (to propositional logic) – means that whole groups of logical laws fall outside the purview of logic. We mean the laws which join propositions with different structures and do not occur among the formulae of certain calculi. Such are the laws joining modal signs and quantifiers, implication signs and relation signs, signs of predication and of class-inclusion, etc. What is more, we find in logic implicit assumptions which can be explicated only by formulating a system of "residual" assertions in order to unify the various sections of logic into a unified, complex logical system. Our presentation tries to take such "residual" laws of logic into account. We would note, in conclusion, that this conception of logic makes possible a more differentiated analysis of logical forms than is usually the case. This happens most particularly in the treatment of the various forms of entailment.

To make our formulations as compact and intuitive as possible, we will use the symbols

$$\cdot, \vee, \because, \sim, \rightarrow, \leftrightarrow$$

in the following sense:

1) $X \cdot Y$ for "X and Y", "Each of X, Y"; $X^1 \cdot X^2 \cdot \ldots X^n$ – "X^1 and X^2 and ... and X^n", "Each of $X^1, \ldots, X^n$"; here and below X, Y, $X^1, \ldots, X^n$ are any sentences;

2) $X \vee Y$ for "at least one of X and Y"; $X^1 \vee X^2 \vee \ldots \vee X^n$ for "at least one of $X^1, X^2, \ldots X^n$".

3) $X\!:\!Y$ for "Either X or Y", "X or Y", "One and only one of X, Y"; $X^1\!:\!X^2\!:\!\ldots\!:\!X^n$ – "Either X^1, or $X^2, \ldots$, or X^n", "One and only one of $X^1, X^2, \ldots X^n$";

4) $\sim X$ for "Non-X", "It is not as affirmed in X";

5) $X \rightarrow Y$ for "If X then Y";

6) $X \leftrightarrow Y$ for "X if and only if Y"; an abbreviation for $(X \rightarrow Y) \cdot (Y \rightarrow X)$.

In the sequel we will be more precise about the signs "and", "or" and "not". For the moment, however, we will assume that their meaning is known to the reader at least to the extent necessary for explaining the matter at hand. The same is the case for the signs, "if... then" and "if and only if". In other words, we assume that the reader already has some skill in handling logical tools, i.e., that he has the logical minimum.

Definitions and assertions will be numbered with the help of *Di*, *Ai* and *Ti*, where *i* is the ordinative numeral of the definition or assertion in a given paragraph; *A* indicates that the sentence is taken as an axiom; *T* indicates that the sentence can be obtained as an inference from axioms. In cross-referencing, the chapter and paragraph numbers will be written after the *i*. For example, *T*3V7 will designate the third theorem of the seventh paragraph in the fifth chapter.

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CHAPTER ONE

THE LOGICAL THEORY OF SCIENTIFIC KNOWLEDGE

1. SCIENTIFIC KNOWLEDGE

Science is a special sphere in man's division of labor, the task of which is the production (obtaining, having) of knowledge and the discovery of new means for it. Scientific knowledge is knowledge had in science.

From the logical point of view scientific knowledge can be distinguished from the extra-scientific (had outside the sphere of science) only by carefully distinguishing its complex forms and methods. This requires professional training and is not found outside of science because of the lack of the requisite habits and intentionality. But science also contains simple forms of knowledge and methods which are hardly distinguishable from those existing outside of it. Therefore, the study of scientific knowledge in the framework of logic is the study of knowledge in general, including forms and methods which are met only in science.

2. BASIC ABSTRACTIONS

The logical investigation of scientific knowledge is based on a series of abstractions and assumptions which limit its possibilities.

Not taken into consideration here are the psychological, social, etc., connections which accompany or influence the obtaining and employment of knowledge. Knowledge is exclusively conceived as information on some object-domain, i.e., as a representation of such a domain. It is assumed that the apparatus of sense-reflection is necessary for the obtaining, preservation and use of knowledge. But its activity is not taken into consideration. No role is played here by that which goes on in the brain or in the organism of man (or within any reflecting being or device). Knowledge is here taken strictly as perceived (seen, heard, etc.) objects of a special type (a special type of thing) and as spatial-temporal structures made up of such objects. We are also not taking into consideration those cognitive aids (instruments, etc.) which complement and strengthen the apparatus of sense-reflection and guarantee the observation of the objects being studied.

The results of knowledge are usually fixed in the sentences of some language. On this basis science forms tools, like formulae, schemata, graphs, tables, which are included in the language of science. Here these are all reduced to sentential form. The point of this abstraction is that every linguistic structure which expresses knowledge is correlated with a set of sentences which adequately express the information in question. This abstraction corresponds to the fact that the man who has to do with science is used to using various linguistic constructs (graphs, tables, etc.) and to "reading" them in the sentences of ordinary language.

Assertions are reduced to a standard form: to the form of sentences (judgements). This abstraction corresponds to the fact that a man engaged in scientific activity knows how to distinguish in each sentence the logically (described in logical terms) structural elements and their mutual disposition; i.e., he knows how to establish the logical structure of the sentence. This abstraction is meaningful in reference to any language but always in reference to some given language. It is here necessary to abstract from those associations expressed in the rules of a language like Russian, English, French, etc., and to take the logical structure of the sentence as something independent of these rules.

3. THREE ASPECTS OF THE INVESTIGATION OF KNOWLEDGE

Scientific knowledge can be viewed from three different angles: structure, construction and meaning. Each of them has its peculiarities which are described in a special system of concepts. In the first of these one observes all the objects which are studied in the logical theory of scientific knowledge, i.e., one articulates the knowledge itself; one lays the perceived out in its parts and into their ordering in space and time. These objects are terms, logical signs like "and", "or", "if... then...", "that, which", "all", different types of structurations of the terms and logical signs, complexes of sentences and terms.

In the second view one distinguishes the means of obtaining knowledge, its parts and combinations. One takes into consideration the activities which make knowledge and its parts possible (induction, deduction, modelling, definition, extrapolation, interpolation, etc.). Since one abstracts from sense-reflection, one considers only the means of obtaining sentences and terms from sentences and terms. Of course, one finds in every branch of science some set of terms and sentences which are obtained without the use of other terms and sentences or which are taken ready-made from other spheres of knowledge. Observation and experiment are important in this context only to the extent that they establish sentences.

One of the biggest problems here is to explain the elementary forms of knowledge by giving an exact account of their properties, to present all knowledge as a construction from the simpler (and, ultimately, elementary) according to more or less general rules, i.e., to describe the standard means for constructing complex forms of knowledge from elementary ones. In the third view, one deals with the relationship of knowledge and its parts to the object-domain, the representation of which it is supposed to be; one tries to find the principles according to which assertions and the structural elements that represent them are accepted in science; in particular, one pays attention to the means of verifying knowledge and to the establishment of the meaning of terms. Here one finds concepts like "meaning", "sense", "true", "false", "exact", "confirmation", which are usually called semantic.

These aspects can to some extent be separated since there is no strict, univocal connection between them. Knowledge can contain elements with the same structure but obtained in different ways: it can contain elements with different structures obtained in the same way. Knowledge can be obtained in one way and verified in another, etc. But a sufficiently detailed and systematic investigation in each of them is impossible without a correlative investigation in the others. The fact is that for the description of the properties of the logical signs found in sentences one has to state the conditions of their use, i.e., to state how one arrives at a sentence with such signs and what can be obtained from it. The concepts, "true", "false", etc., are defined differently for different types of sentences: some sentential structures cannot be distinguished without reference to the method of construction or verification. To each structure of sentences and terms corresponds some set of methods which makes it possible to obtain sentences and terms with the structure in question. Description of the basis for acceptance of the terms and sentences into science is a retrospective description of the possible methods of constructing them since the matter of meaning is – in a sense – a semantic periphrasing of the matter of construction. In brief, the unity of these aspects is a necessary condition for the investigation of scientific knowledge: it is not just an assumption.

4. INTUITION

To study scientific knowledge is, above all, to study the practical habits which people have in the history of knowledge built up for obtaining and working with knowledge (and which the investigator has somehow acquired in the course of his individual formation). These habits are not given by nature. They are formed by people and are reformed in the course of scientific progress. Those who have these habits have a (more or less clear and defined) practical or intuitive understanding of the properties of knowledge. This understanding of knowledge is necessary to the very habits of operating with it. Formulation of it is the starting point of logic as a special science and constitutes the line of continuity between its first results and the cognitive activities of people.

The intuitive understanding we are talking about here arises spontaneously with all the resultant lack of clarity, structure, completeness, etc. And logic has to exert itself in order to make it clear and unambiguous, to standardize, to explicate, etc. And this is not a simple description of something well known and accepted. This is the continuation of the spontaneous activity of people in the forming and perfecting of the logical modes of language, but already at a professional level. From the outset, logic establishes something new compared to that which is known in intuition.

It follows that the logical theory of scientific knowledge has to take account of the intuitive understanding of some aspect of the cognitive activity of people, but cannot become its slave. It has to assume the possibility of leaving the realm of intuition. And one of the tasks is explaining how and to what extent this departure can be effected.

From what has been said it also follows that the notion "approximating logic to natural language" is the result of a misunderstanding.

5. LOGICAL CALCULI

Logical calculi (formal constructions) occupy an important place in contemporary logical theory of scientific knowledge. There are two uses for logical calculi. First, there is the explication of some elements of intuition: and intuitive understanding of some types of knowledge is accompanied by a logical calculus and interpretation which establishes their correspondence. If there is no immediate correspondence, then the calculus is either adapted to the intuitive premisses by the introduction of complements, limitations, etc., or one reconstructs it in such a way as to obtain the correspondence. This method of theoretical construction provides results only for isolated problems since the results are partial and one is satisfied with "paradoxical" (not corresponding to intuitive understanding) consequences; but we are not denying their cognitive value (the possibility of the use of deduction and prediction, provability, explication of concepts, exclusion of ambiguity, simplicity, etc.).

In a second view, the logical calculus is considered as independent of intuition, i.e., it is seen as something formed by the logician as a complement to the logical methods already formulated in science. In both cases there is a set of problems which belong to the logical theory of scientific knowledge.

Thus, the logical theory of scientific knowledge is not a special science, different from and parallel to logic. It is simply that part (or aspect) of logic which makes it possible for the formal apparatus of logic to be counted precisely as an apparatus of logic and not of any other science. It explains the basis on which this apparatus is set up, the direction of development of it, and paths of using it in the describing and perfecting of the language of science.

6. ORDINARY LANGUAGE AND SCIENTIFIC LANGUAGE

The language of science is the empirical given, the observation of which provides the point of departure for the logical theory of scientific knowledge. In turn, this language of science is based on ordinary language: the destruction of ordinary language would involve the destruction of the language of science (it would become incomprehensible).

The limits between scientific language and ordinary language are relative and historically conditioned. Some terms and sentences of scientific language find their way into ordinary language. On the other hand, many of the terms and sentences of ordinary language are used in science. One uses ordinary language both for the introduction of the special terms of science and for the explication of scientific sentences. Modes for constructing terms and sentences in ordinary language can be used for certain purposes in scientific language, etc. However, the distinction of scientific language as a superstructure over ordinary language has sense as an abstraction in the framework of logic. This is because ordinary language is formed and learned as an element in a very complex situation which includes the evolution of humanity and of each individual man. The accumulated knowledge and the methods of accumulation can be only imperfectly described in terms of logic. Assuming here that ordinary language is given, we thereby assume certain terms, sentences, methods as not subject to further logical analysis (we assume some sort of "prelogical" or "extra-logical" means of obtaining knowledge).

Therefore, the logical theory of scientific knowledge is limited not only "from above" (reduction of any forms of knowledge to sets of sentences) but also "from below": it leaves aside all the means and conditions of knowledge which are involved in operating with ordinary language and which are not subject to description in logical terms.

7. OBJECTIVITY OF APPROACH

The investigator's activity can be seen: 1) subjectively, i.e., as the investigator lives it; 2) objectively, i.e., as it can be observed (describing only that which can be seen, heard, etc.).

The subjective approach was once widespread in logic. This was psychologism in logic, and no longer has any importance. But it shows its head from time to time especially in discussion of questions which go beyond the purely formal apparatus of logic. It is very difficult to rid oneself of it completely since every normal man has the ability to examine himself and is convinced of the existence both in himself and in others of some "inner", "spiritual", "ideal", etc., life.

We here take a purely objective view of knowledge. It is evident that the concepts, "ideal", "spiritual", "conceptual", etc., which are usually used in the subjective approach, lose any practical importance since knowledge itself is being taken as something tangible. Even in those cases where the objects of knowledge are intangible and not subject to observation, knowledge itself has to be subject to observation. Otherwise, there would be no knowledge.

The objective approach will here mean that we (author and reader) will assume some investigator (i.e., someone who has knowledge and operates with it) whose cognitive activity we can observe and, within limits, control. This "within limits" means that we will assign to the investigator certain properties and capacities and then we will observe what he must do in order to solve some problem of knowledge. In what follows references to the investigator will be dropped for reasons of style; but they could easily be reinstated. As investigator, one could have in mind not only a man engaged in scientific activity but in general any being or mechanism capable of accomplishing what is assumed in each case.

CHAPTER TWO

SIGNS

1. OBJECT

We will use the word "object" in its widest sense: an object is anything which can be perceived, represented, named, etc.; i.e., anything at all. Objects will be represented by the symbols

$$\Pi, \Pi^1, \Pi^2, \Pi^3, \ldots .$$

Each of these symbols in isolation will designate any (indiscriminately) or every (this will be clear from the context) object. Distinctions between symbols occurring together will only mean differences between objects (and not necessarily the presence or absence of some perceptible or assumed properties; it could be that the distinction is only one of time and place). The ability to distinguish and correlate objects is taken for granted.

2. SELECTION

We will consider that the investigator has selected an object if in some way "he adverts to it". There are two actions here:

1) he establishes or reproduces a sense image of the object (he sees, hears, imagines, etc., it), uses its name, says something about it, establishes or studies its schema, outline, photograph, etc.;

2) he carries out some further activity which reveals or confirms that at a certain time and for a certain reason he turned his attention to that object to the exclusion of all others, i.e., during that time he gave it some sort of priority over the others.

We consider selection of the object to be an elementary activity in any cognitive process. It is not analyzed within the framework of the logical theory of scientific knowledge. The term "selection" is taken as primitive, explained only in terms of ordinary language and by examples. Thus the investigator selects the electron, indicating its effects on a photographic plate; he selects phlogiston by affirming in some context that phlogiston does not exist; he selects a triple of numbers x, y and z by considering the formula, $x^2+3xy=2z$, etc. The selection of an object is always localised in time: the temporal interval when a given object is said to be selected by a given investigator is always more than zero. This means that the selection of an object under all circumstances (even if these do not come to be) is some state of the investigator; more precisely, it is a state of his natural apparatus of reflection.

3. COMPARISON

If the investigator selects two or more different objects, we will say that he compares these objects (or effects their comparison). The objects compared can be selected simultaneously or one at a time. But there is always an interval of time when they are all considered selected by the investigator who effects their comparison; the act of selection is localised in time. Evidently, the comparison of objects is an aggregate made up of two or more different acts of selection, which are in some order. For example, in constructing the sentence, "Water is formed by uniting hydrogen and oxygen", the investigator has selected the objects, water, hydrogen and oxygen; the ordering of the acts of selection is expressed in the sequence of their names in the sentence; the localisation of the acts of selection is expressed in the fact that a sentence is constructed which relates to the three objects selected and which is experienced as a whole. The objects compared can also be selected independently of each other.

4. CORRESPONDENCE

D1. We will say that the investigator has established a correspondence of object Π^2 to object Π^1 (or that object Π^2 corresponds to object Π^1) if and only if the following is the case: each time that the investigator selects Π^1 he thereupon selects Π^2, being put before the alternative of choosing or not choosing Π^2. The correspondence of the object Π^2 to the object Π^1 will be expressed as

$$\Pi^2 \Leftarrow \Pi^1,$$

and its absence as

$$\sim(\Pi^2 \Leftarrow \Pi^1).$$

The expression "thereupon" in *D1* simply means that the selection of Π^2 happens after that of Π^1. The expression "being put before the alternative..." can be explained as: 1) we set up in a certain way some set of objects among which Π^2 is found and we have the investigator choose any one of them; this is done several times; the proposed set can be varied; 2) we can assume that the investigator, simultaneously with the selection of the object, will carry out some action confirming that the selection was done, compelling the investigator to select Π^2 and to experience the confirmation of the activity.

A correspondence established between two objects does not mean that they will always be selected together. Each of them can be selected independently of the other (without the selection of the other). What is more, if such a possibility is lacking there can be no talk about any correspondence at all.

We have defined the simple case of correspondence. Through it, other forms are defined:

D2. Π^1 and Π^2 are in mutual correspondence if and only if

$$(\Pi^1 \Leftarrow \Pi^2)\cdot(\Pi^2 \Leftarrow \Pi^1).$$

D3. Π^2 univalently corresponds to Π^1 if and only if

$$(\Pi^2 \Leftarrow \Pi^1)\cdot \sim (\Pi^3 \Leftarrow \Pi^1),$$

where Π^3 is any object different from Π^2.

D4. Π^1 and Π^2 are in one-to-one correspondence with each other if and only if the first univalently corresponds to the second and vice versa.

If

$$(\Pi^2 \Leftarrow \Pi^1)\cdot(\Pi^3 \Leftarrow \Pi^1)$$
$$(\Pi^2 \Leftarrow \Pi^1)\cdot(\Pi^2 \Leftarrow \Pi^3)$$
$$(\Pi^2 \Leftarrow \Pi^1)\cdot(\Pi^3 \Leftarrow \Pi^1)\cdot(\Pi^2 \Leftarrow \Pi^4),$$

is the case then one talks about one-many, many-one, many-many correspondence, respectively.

We should note that $\Pi^2 \Leftarrow \Pi^1$ establishes conditions for the selection of Π^2 following on that of Π^1. This does not mean that the same conditions are involved in the case of $\Pi^1 \Leftarrow \Pi^2$; other conditions have to be met (this is evident from the fact that the order of selection of objects is different). Nor does this mean that the conditions are met in the case $\Pi^3 \Leftarrow \Pi^2$ or $\Pi^1 \Leftarrow \Pi^3$.

It is clear from the definitions that correspondence is the ability of the selector to carry out certain activities, i.e., to choose strictly determined objects as a consequence of choosing other objects, when the conditions for selection are fulfilled. The principle of transitivity will not hold for correspondence since it has the conditional form, "If one selects Π^1 and it is necessary to select an object from some set of objects, then one will select Π^2".

From the viewpoint of correspondence objects are taken as unchanging and as not influencing one another (rather, their changes and mutual influences are not taken into consideration). Correspondence is usually established between structures and not between objects which are influencing each other. Otherwise it would be insignificant and impractical.

$\Pi \Leftarrow \Pi$ is excluded since by definition correspondence requires two different objects. If the objects Π^1 and Π^2 are not distinguished by the investigator as samples of objects of one and the same type, they are distinguished by their position in space or time. Otherwise, the notion of correspondence would lose any practical significance.

The correspondence of objects has nothing in common with causal connections between objects. The causal connection of objects does not depend on its being known by an investigator while correspondence would not exist without the investigator's knowing the objects (i.e., without the investigator's will-act; correspondence is his property). In the case of the causal connection one is interested in the dependence of the existence and occurrence of the properties of certain objects on the existence and occurrence of the properties of other objects. In the case of correspondence this is excluded to the extent that it is necessary for the identification (designation) of the objects. The investigator can establish a correspondence between objects which are causally connected and he can find causal connections between objects in correspondence. But this does not change what was said above.

Correspondence results from the investigator's resolve to consider that one object corresponds to another (and to act in accord with this resolve), from a spontaneously formed habit, from a necessity imposed by other investigators, etc. But in all instances this is the formation in the investigator of the ability to carry out certain actions – and nothing more.

5. SIGN

D1. If the investigator specially uses (establishes, forms, produces) Π^1 for the sake of its reciprocal correspondence with Π^2, then Π^1 will be called the sign of Π^2 and Π^2 will be the designatum of Π^1. We will also use the expressions, "Π^1 designates object Π^2" and "Π^2 is designated by the sign Π^1".

Signs will be represented by the symbols

$$З, З^1, З^2, З^3, \dots .$$

Each of these symbols in isolation will represent any sign and a difference between symbols used together is a difference between signs.

Signs are distinguished or not distinguished by their physical, i.e., perceptible, form. If signs are considered physically identical, they are samples (repetitions, reproductions) of one and the same sign (*D2*).

It follows from the definition of "sign" that if some object is a sign then one can select some other objects that will be in reciprocal correspondence with it. Unless otherwise indicated, we will assume that for the signs $З, З^1, З^2, \dots$ the objects $\Pi, \Pi^1, \Pi^2, \dots$, respectively are of such a nature.

The formation of a sign (i.e., whether some object is a sign or not) depends entirely on a will-act of the investigator. And if there is more than one investigator, the decision to call an object a sign will have to be agreed to by the others.

Signs have properties other than that of being in correspondence with a designatum. Not just any object is suited to serve as sign. Only certain types of objects are professional signs. And as signs they conserve their place and role in correspondence. Signs have to be directly perceptible to those for whom they are intended. The notion of correspondence implies that signs are invariable in their function as signs.

Objects become signs not because of some causality within them but because the investigator so decides. Signs differ from the sensible images of objects: the latter are states of the investigator, i.e., states of his natural reflective apparatus, while the former are objects existing outside of and independent of the investigator. They play a definite role in the life and activity of the investigator: they are created and used by him: but they are not his internal states. The set of signs and rules for operating with them makes up the semiotic (or artificial) apparatus of reflection. It is obvious that this is impossible without the natural (sensible) apparatus of reflection.

From the definitions of "correspondence" and "sign" it follows that an object cannot be a sign of itself. But there are cases where the distinction between the designatum and the signs is none too clear. This happens especially when signs are themselves taken as special types of objects (i.e., not as signs) and are designated by signs. Less serious are cases where as sign for objects of a certain type one takes representatives of this same type (e.g., as a sign for the numbers 1, I, "one", "unit", etc., one can take any one of these so that the number itself becomes a copy of its own sign). We will assume that the difference between signs and designata can be strictly established in all instances.

6. VALUE OF THE SIGN

D1. Let З be the sign for П. The value of З will precisely consist in its designation of П. In other words, in answer to the question as to what the value of the sign З is, the investigator has in some way to indicate to us precisely what (which object) this sign designates.

The value of З is not the object П, nor the "thoughts" which might appear in the head of the investigator during his operation with З, but it is the fact that it designates П, and the investigator knows this.

To the question "what is the value of З?", the investigator can respond in various ways: reference to sense-perceptible objects, descriptions in words, representation in gestures, etc., or construction of concepts, indications on the rules of operating with the sign in different situations (contexts), etc. But all this has to do with ways of establishing the value of З and not with the definition of the meaning of the expression "the value of the sign З".

A sign has meaning for a given investigator if he can somehow select from a set of objects (either sensibly or through description in terms of other signs with meaning) at least one which corresponds to this sign. If he cannot do this, the sign has no meaning for him. And it is thus no sign at all. The expression, "the sign has no meaning" is equivalent to "that which the investigator takes to be a sign is not a sign", and the expression, "the sign has meaning" is the same as "this is a sign".

Such expressions figure here only to the extent that objects of a certain type are introduced as signs. The habit of connecting the term "sign" not only with the function of objects but also with their perceptible form leads to using it for sign-like objects (e.g., lines on paper, sounds, etc.).

There are cases where one and the same object like 3 is the sign for some objects from the point of view of some investigators and the sign for other objects from the viewpoint of other investigators. In such cases one talks about the "multivalence" of the sign. We exclude such cases, i.e., we assume: a sign has one and only one value; for the above mentioned cases we use different signs; in principle, one can always determine that 3 plays the role of different signs for different investigators and one can eliminate this by introducing differing $3^1, 3^2, \ldots$. If 3 designates Π^1, $\Pi^2, \ldots$ this does not mean that it is "multivalent": its value is such that it designates $\Pi^1, \Pi^2, \ldots$.

7. RELATIONS BETWEEN SIGNS

D1. The sign 3^1 is included according to value in 3^2 if and only if any object designated by 3^2 is designated by 3^1. We will abbreviate this as

$$3^1 \rightharpoondown 3^2.$$

We will write negation as

$$\sim(3^1 \rightharpoondown 3^2).$$

D2. The signs 3^1 and 3^2 are identical in value if and only if

$$(3^1 \rightharpoondown 3^2)\cdot(3^2 \rightharpoondown 3^1).$$

We will abbreviate this as

$$3^1 \rightleftharpoons 3^2.$$

We will write negation as

$$\sim(3^1 \rightleftharpoons 3^2).$$

D3. The value-range of 3 is the set of all possible signs which include it according to value. In other words, if $3 \rightharpoondown 3^i$, then 3^i is an element of the value-range of 3.

A1. $(3^1 \rightharpoondown 3^2)\cdot(3^2 \rightharpoondown 3^3) \to (3^1 \rightharpoondown 3^3)$.

A2. If every element of the value-range of 3^2 is an element of the value-range of 3^1, then $3^1 \rightharpoondown 3^2$.

Consequences of *A1*, *D2* and *D3*:

T1. $3 \rightharpoondown 3, \quad 3 \rightleftharpoons 3$

T2. $(3^1 \rightleftharpoons 3^2) \cdot (3^2 \rightleftharpoons 3^3) \rightarrow (3^1 \rightleftharpoons 3^3)$

T3. If $3^1 \rightharpoondown 3^2$, then every element of the value-range of 3^2 is an element of the value-range of 3^1.

We will examine other relations between signs below when we take up terms as special cases of signs.

8. SIMPLE AND COMPLEX SIGNS

Some signs are formed from others by joining the others with fields of a special type in some standard way. We will call such fields sign-generative operators (*D1*).

D2. We will call signs structurally complex (simple) if they are analyzed (not analyzed) into other signs and sign-generative operators.

Structurally complex signs will be represented by the symbols

$$\{\alpha; 3^1, \ldots, 3^n\},$$

where $3^1, \ldots, 3^n$ $(n \geqslant 1)$ are signs and α means that this sign is constructed with the help of some operators. If $n=1$, then the complex sign will have the form $\{\alpha; 3\}$

A1. $\sim (3^i \rightharpoondown 3^k) \rightarrow \sim (3^l \rightleftharpoons \{\alpha; 3^1, \ldots, 3^n\})$,

where $i=1, \ldots, n$; $k=1, \ldots, n$; $i \neq k$; $l=1, \ldots, n$.

$$\sim (3 \rightharpoondown \{\alpha; 3\}) \cdot \sim (\{\alpha; 3\} \rightharpoondown 3).$$

In the formation of complex signs from simple ones there is a change of the latter such that one needs some skill in order to discover out of which signs a given complex sign is constructed. We assume the presence of such skill: this is equivalent to assuming that a complex sign is a set of strictly localized signs, ordered in time and space. If in the process of combining the type of sign changes so that there is no physical resemblance with the point of departure, then there have to be conventions on the relationship of the meaning of the primitive signs and their modifications in the context of the complex sign as physically distinct signs.

Simple signs are combined into complex ones according to some kind of rules and there is something in the complex sign which refers to them:

this is the proximity and order of the signs in space and time and also some complementary objects, forming with the combined signs some physical whole, i.e., sign-generative operators. We presuppose the presence of habits of operating with them. (We assume that their properties are known.) We also assume that if there is a case where only one spatial-temporal disposition of simple signs is enough for the formation of a new sign, then there will always be found sign-generative operators which play the same role.

Signs formed from other signs can be divided into two groups:

1) signs, the value of which is known if the value of the component signs is known,

2) signs, the value of which cannot be determined if one only knows the values of the component signs. In both cases we assume that the rules for combining signs are known. For example, both "kilogramometer" and "dynamometer" are composed of two different words. But the first indicates the result of some operators of measuring magnitudes while the second is a device for measuring magnitudes. These meanings cannot be established if one knows only the meanings of the component parts and the rules of combination.

Thus, one has to distinguish:

1) rules for combining signs into new signs, which do not depend on the particularities of any signs as material bodies and which make it possible to obtain signs of the first group;

2) rules for combining signs as special material bodies (sounds, lines on paper, etc.).

The examples cited above are regularly constructed in English according to the rules of the second group but they are not signs according to the rules of the first group.

With this in mind we expand *D2*:

1) to establish the value of a complex sign it is enough to know the values of all the simple signs it contains and the properties of all of its operators;

2) if some sign cannot be given a value in this way it has to be considered structurally simple.

D3. 3^1 depends as to value on 3^2, if and only if it is necessary to know the value of 3^2 for the establishment of that of 3^1.

T1. It follows from *D3* that $\{\alpha; 3^1, \ldots, 3^n\}$ (where $n \geqslant 1$) depends as to value on $3^i\,(i=1,\dots,n)$. The sign which can be formed from 3^1 by substituting 3^3 for 3^2, will be represented by

$$3^1(3^2/3^3).$$

As abbreviation for

$$((3(3^1/3_1))\dots)(3^n/3_n)$$

we will use

$$3(3^1,\dots,3^n/3_1,\dots,3_n).$$

A2. $(3^1 \rightleftharpoons 3^2) \rightarrow (3^3 \rightharpoonup 3^3(3^1/3^2))$

T2. $(3^1 \rightleftharpoons 3^2)\cdot(3^3 \rightleftharpoons 3^4) \rightarrow (3^1(3^3/3^4) \rightleftharpoons 3^2(3^3/3^4))$

T3. $(3^1 \rightleftharpoons 3^2) \rightarrow (3^3 \rightleftharpoons 3^3(3^1/3^2))$.

9. MEANING OF THE SIGN

D1. The meaning of a simple sign is its value; the meaning of $\{\alpha; 3^1,\dots,3^n\}$ consists in the fact that it is constructed from the signs $3^1,\dots,3^n$ with the help of operators α and their meaning is known.

T1. A structurally simple sign has (does not have) meaning if it has (does not have) value; a structurally complex sign has meaning if every one of its component signs has value (rules of construction have not been violated) and it does not have meaning if at least one of its component signs does not have value. In other words, the investigator knows the meaning of a sign if and only if he knows the values of all the simple component signs and the properties of all the operators.

The identity of meaning of signs 3^1 and 3^2 will be represented as

$$3^1 \equiv 3^2,$$

and its absence as

$$\sim(3^1 \equiv 3^2).$$

The identity of meaning of signs is defined by the assertions:

A1. If 3^1 and 3^2 are structurally simple signs, then

$$(3^1 \equiv 3^2) \leftrightarrow (3^1 \rightleftharpoons 3^2)$$

(structurally simple signs are identical in meaning if and only if they are identical in value).

A2. $(3^1 \equiv 3^1(3^2/3^3)) \leftrightarrow (3^2 \equiv 3^3)$.

According to *A2* the question about the identity of meaning of two given signs reduces to that about the identity of meaning of the component simple signs. This assumes that both signs are constructed according to the same logical rule.

T1. $(3^1 \equiv 3^2) \rightarrow (3^1 \rightleftharpoons 3^2)$
T2. $\sim((3^1 \rightleftharpoons 3^2) \rightarrow (3^1 \equiv 3^2))$
T3. $(3^1 \equiv 3^2) \rightarrow (3^3 \equiv 3^3(3^1/3^2))$
T4. $(3^1 \equiv 3^2)\cdot(3^2 \equiv 3^3) \rightarrow (3^1 \equiv 3^3)$.

Assertions *T1* and *T2* mean that signs identical in meaning are identical in value but not always vice versa. For example, in defining the structurally simple sign 3^1 through the structurally complex sign 3^2 we come to consider 3^1 and 3^2 identical in value but their meaning is not identical since one is simple and the other complex.

Paradoxes of the evening-star type are the result of confusing different signs. If we know only that these expressions are complex signs made up of the signs "evening", "morning" and "star", then they are different in meaning (provided, of course, that the signs "evening" and "morning" are different in meaning). But the question as to their value remains open.

On the other hand, if these expressions are deliberately being used as different designations of one and the same object, then we have to do with different signs: they now are structurally simple signs, identical in value (and, therefore, in meaning).

10. CONSTRUCTION OF SIGNS

D1. When the value of a sign is established (the sign is created) without the use of other signs we will call it simple in construction or primitive. If, however, the value of the sign is established through use of other signs (even just one) we will call it complex in construction (or derived). It is clear that a sign which is simple in construction is structurally simple and one which is structurally complex is complex in construction. But the latter can be structurally simple since there is not a full coincidence of the planes of structure and construction. The words "kilogramometer" and "dynamometer" are simple in structure but complex in construction: their value is explicated with the help of other signs ("instrument", "measure", "magnitude", etc.). Signs which are complex in construction are formed by agreement on the relations of the signs to the newly introduced signs, according to value. More about this below.

D2. When the value of a sign which is complex in construction can be established without using some of the signs which are used in the establishment of its value this sign is called analytic: otherwise, it is synthetic.

11. CATEGORIES OF SIGNS

Signs are classified into categories in such a way that the following hold:

A1. If a sign belongs to a certain category, it belongs to no other.

A2. If 3^1 and 3^2 belong to different categories, then $\sim(3^1 \rightharpoonup 3^2)$ and $\sim(3^2 \rightharpoonup 3^1)$.

T1. If 3^1 and 3^2 are signs of different categories, then $\sim(3^1 \equiv 3^2)$.

Special operators and combinations of signs are needed to convert signs of one category into those of another. Sign-generative operators can now be classified as applicable to signs of one and the same category and to signs of different categories, and as providing signs of one category and of another, etc. A general theory of signs, which can be constructed independent of the interests of logic (i.e., as a special discipline), has to take account of all possible, logically conceivable cases of this type as well as of the corresponding assertions.

12. EXISTENCE OF OBJECTS

In dealing with signs it is important to know if the objects which correspond to them exist or not.

It is impossible to find a definition of existence and non-existence which would satisfy all sciences and all instances of knowledge. There are in different sciences and in different sections of the same science different notions of existence and non-existence. Instead of clear definitions one usually finds vague conventions. Normally existence and non-existence are understood as the possibility or impossibility of observing objects with the sense-organs and with instruments, detecting their traces and effects, and also as the possibility or impossibility of creating objects of the type in question. In some cases the existence or non-existence of some objects is explicitly or implicitly postulated and the question about the existence or non-existence of the others is resolved by inference from these premisses.

But there are elements common to all sciences:

1) in every domain of science there is at least one mode of selection of at least some of the objects studied, which differs from the selection of these objects by means of the simple use of the signs designating them and to which can be attributed the property that if selection of the object by this mode is possible (impossible), then it is existing (non-existing). Such a mode of selection is called existential (*D1*): it is relative to such existential selections that one defines the expressions "exists" and "does not exist" and that one constructs sentences where such expressions appear;

2) for each of these domains of science one can draw up a list of rules which make possible a judgement on the existence or non-existence of other (at least some) objects, on the basis of the information gleaned in the first point: one cannot talk here about all the other objects investigated in the science in question since there are cases where the question of the existence or non-existence of the objects is unsolvable (on the basis of the premisses at hand);

3) the existential selection of objects assumes some sort of determined (in some way or other) domain of space and time: the same is true of conditions (in particular, of the system of definitions and assumptions); and in every case this has to be known.

For example, let us take the expressions, "A set of three integers, x, y and z, such that $x^2+y^2=z^2$" and "A set of three integers x, y and z, such that $x^3+y^3=z^3$". Use of these expressions is selection of the corresponding triples. But there is another mode of selection: to write the numbers by means of the signs of natural numbers or to indicate a means of doing this in a finite number of steps. Relative to this second mode of selection a triple designated by the first expression exists; that designated by the second does not. When discussing the existence of Peter I, one talks not about his living in such and such a time, but in a historical sense. And the indication of this existence (existential selection) is the written evidence.

D2. If 3 is a sign for Π and Π does not exist (exists), then 3 is an empty (non-empty) sign. An empty sign has meaning and value.

CHAPTER THREE

TERMS

1. TERMS

Terms are the signs which make up sentences. These signs have certain physical properties which suit them for this role: ease of construction and perception; general availability; unlimited repeatability, etc. We assume all these properties to be given: i.e., we make the following assumption: in every branch of science one knows the properties signs must have in order to be terms. In other words, we assume that relative to a set of objects it is known that they are terms. The task now becomes to study the rules for forming from them new terms and for forming sentences from terms. Terms will be designated by the symbols

$$t, t^1, t^2, \ldots.$$

Everything that was said about signs in general applies to terms. Below we will present a series of definitions and theorems relative to terms. But, from them one can obtain theorems and definitions for signs in general by replacing the word "term" with the word "sign". This means that we regard terms exclusively as signs. The fact that these are signs of a definite physical form plays no role in the exposition undertaken below. We will therefore not introduce a strict definition of "term". We will limit ourselves to the assumption: terms are the signs which are the elements of the language of science.

D1. Term t^1 is called general (generic) relative to t^2, and t^2 is particular (specific) relative to t^1 if and only if

$$(t^1 \rightharpoonup t^2) \cdot \sim (t^2 \rightharpoonup t^1).$$

D2. Term t is called individual if and only if it is impossible to have a t^i such that

$$(t \rightharpoonup t^i) \cdot \sim (t^i \rightharpoonup t)$$

(i.e., if and only if it cannot be generic).

D3. Term t is called maximally (in the limit) general if and only if it is impossible to have a t^i such that

$$(t^i \rightharpoonup t) \cdot \sim (t \rightharpoonup t^i)$$

(i.e., if and only if it cannot be specific).

D4. Two terms t^1 and t^2 are compatible in value if and only if it is possible to have a t^3 such that

$$(t^1 \rightharpoonup t^3) \cdot (t^2 \rightharpoonup t^3).$$

D5. The terms t^1 and t^2 are comparable if and only if there is possible a t^3 such that

$$(t^3 \rightharpoonup t^1) \cdot (t^3 \rightharpoonup t^2).$$

D6. The division of t is the set of all possible terms $t^1, \ldots, t^n$ from the value-range of t, which are incompatible in value, where $n \geqslant 2$; the terms $t^1, \ldots, t^n$ are elements of the division of t.

D7. The extension of t is the set of all possible individual terms from the value-range of t; t^i is an element of the extension of t if and only if it is an individual term from the value-range of t.

The following consequences follow from *A1*II7, *A2*II7 and the definitions:

T1. If $t^1 \rightharpoonup t^2$, then t^1 and t^2 are compatible in value.

T2. If t^i is an element of the division of t, then $\sim (t^i \rightharpoonup t)$.

T3. Individual terms do not have a division.

T4. If t^i is an element of the value-range of an individual term t, then $t \rightleftharpoons t^i$ (i.e., the extension of the individual term is "equal to one").

T5. If $t^1 \rightharpoonup t^2$ and t^3 is an element of the extension of t^2, then t^3 is an element of the extension of t^1.

T6. If every element of the extension of t^2 is an element of the extension of t^1, then $t^1 \rightharpoonup t^2$.

The terms "object", "one (any) object", "other object", etc., are maximally general, where the words "one" and "other" mean only that the objects can be distinct (but the distinctions between the objects are not fixed). They will be represented by the symbols

$$t^*, t^{*1}, t^{*2}, \ldots.$$

For any t by definition

T7. $t^* \rightharpoonup t, \quad t^{*i} \rightharpoonup t.$

It is possible to have empty individual terms, e.g., "Zeus". It is also possible to have terms to which correspond only one object at a given time but which are not individual, e.g., "a cosmonaut whose first name is 'Yuri'".

2. DEFINITIONS

T1. In asserting that t is a term we (on the strength of the definitions accepted) assume:

1) if t is a simple term, its value is known;

2) if t is $\{\alpha; t^1, \ldots, t^n\}$, the values of all the terms $t^1, \ldots, t^n$ are known.

Thus if t is a term then it does not contain terms whose values are not known (are not taken as given).

D1. To form term t means to make an object having the form of t play the role of a term. Thus, formation of a term is not the forming of the body t, which offers no problem on our assumptions, but the attribution of a definite function (role) to this body. Since this depends on the will and desire of the investigator there is always some agreement or decision (that this can be required by certain circumstances does not change matters). The word "decision" is more to the point. The word "agreement" is *à propos* when other investigators are involved.

D2. Formation of a term by agreement on the relation between its value and the values of other terms will be called definition of the term. One finds definitions of the following types.

Definitions of type I (simple definition): object t^1 (having the form t^1) will be a term (will be considered a term; the investigator will consider it a term) such that

$$t^1 \rightleftharpoons t^2,$$

where t^2 is a term. In short:

$$t^1 = Df.\, t^2.$$

Before the construction of the definition t^1 is not a term.

Definition I is used when t^2 is a complex term. t^1 is here introduced as an abbreviation. Otherwise, this definition has no practical sense.

t^1 is called the definiendum and t^2 the definiens (*D3*). If t^2 is a complex term, then it is obvious that t^1 and t^2 are not identical in meaning. The term defined in I is always simple.

Definitions of type II (definition through enumeration; inductive, recursive definition): object t will be a term such that X, where in X are enumerated all terms t^i such that

$$t \rightharpoonup t^i,$$

and it is indicated that

$$(\sim (t^i \rightharpoonup t^k) \to \sim (t \rightharpoonup t^k)),$$

where t^k is any term distinct in value from every t^i.

Definition II can be divided into two groups. Definition II^1: object t will be a term such that

$$(t \rightharpoonup t^1) \cdot \ldots \cdot (t \rightharpoonup t^n) \cdot (\sim (t^i \rightharpoonup t^k) \to \sim (t \rightharpoonup t^k)),$$

where $n \geqslant 2$. Definition II^2 is more complex and can contain an infinite number of t^i. It has such a form. Object t will be a term such that:

1) $(t \rightharpoonup t^1) \cdot \ldots \cdot (t \rightharpoonup t^m)$,

2) $((t \rightharpoonup t_1^1) \cdot \ldots \cdot (t \rightharpoonup t_1^k)) \to ((t \rightharpoonup t_2^1) \cdot \ldots \cdot (t \rightharpoonup t_2^l))$,

where $m \geqslant 1$, $k \geqslant 1$, $l \geqslant 1$, and $t_2^1, \ldots, t_2^l$ are terms formed from $t_1^1, \ldots, t_1^k$; and, possibly, from other terms;

3) $\sim (t^j \rightharpoonup t^n) \to \sim (t \rightharpoonup t^n)$, where t^j is any of $t^1, \ldots, t^m, t_2^1, \ldots, t_2^l$, and t^n is any other term, differing from them.

In definition II the defined term t is also simple. The distinction between definiendum and definiens is not as literal here as it is in case I.

Definitions of type III: objects t^0 and $\{\alpha; t^0, t^1, \ldots, t^n\}$ will be terms such that

$$\{\alpha; t^0, t^1, \ldots, t^n\} \rightleftharpoons \{\beta; t^1, \ldots, t^n, t_1, \ldots, t_m\},$$

where $n \geqslant 1$, $m \geqslant 1$, α and β can be different or identical. Both terms introduced here are new. Definition IV is definition II for the term $\{\alpha; t^0, t^1, \ldots, t^n\}$, in which we find the newly introduced term t^0. A more detailed description of the properties of the definition requires a description of the properties of operators, which – in turn – requires consideration of the concrete forms of sentences.

The other operations for the formation of terms from given terms are derived from operations on sentences.

T1. If t^1 is defined in such a way that t^2 occurs in the definiens, then t^1 depends as to value on t^2 (according to *D3*II8).

Definitions with variables form a special case of definitions of the type II. They have the form: b will be a term such that X, if and only if $a^1, \ldots, a^n$ ($n \geqslant 1$) are terms such that Y (where X is an assertion containing some b, and Y is an assertion containing $a^1, \ldots, a^n$). $b, a^1, \ldots, a^n$ are here variables with terms as value-ranges. The rule for such definitions is: in the definition itself and in its implications one cannot put for $a^1, \ldots, a^n$ b and all the terms which depend on it (they contain b or are defined with the use of b). This rule is a result of the definition itself, where $a^1, \ldots, a^n$ have to be terms which are independent of the definition of b, i.e., b is not included among them.

These types of definition could be called simple (or independent). Complex (or dependent) definitions simultaneously define two or more different terms, with one of them being used in the definiens of the others.

3. TRADITIONAL RULES OF DEFINITION

Traditional logic required of definitions 1) adequacy, 2) absence of tautology, 3) absence of circularity. Since only definition of type I was considered, these requirements were elucidated only for the simplest case. We will extend them to the other types of definition and will show that they are consequences of definitions and assertions made above.

Let us take definition I. If $t^1 = Df.\, t^2$, then $t^1 \rightleftharpoons t^2$, and, according to *T5*III1, their extensions coincide (satisfying the requirement of adequacy). If t^2 is a term then, by the very definition of definition *D2*III2, t^1 does not occur in t^2. And this means that there is no tautology in $t^1 = Df.\, t^2$. Let t^2 be $\{\alpha; t_1, \ldots, t_n\}$ and $t_1 = Df.\, \{\beta; t^1, t^i, \ldots, t^k\}$; according to *T2*II8, we have: $t^1 \rightleftharpoons \{\alpha; \{\beta; t^1, t^i, \ldots, t^k\}, \ldots, t_n\}$; thus, in order to find the meaning of t^2 it is necessary to know that of t^1 and the latter is not a term in the definition at hand. If a term which occurs in t^2 is defined through t^1 then the definition of t^1 through t^2 becomes impossible. This meets the requirement of non-circularity.

The requirement of adequacy for II^1 is obvious in view of $\sim(t \rightarrow t^k)$: all elements of the extension of $t^1, \ldots, t^n$ are, according to *T5*III1, elements of the extension of t and there are no other terms in which t is included according to meaning. On the second requirement the situation is the following: if $n=1$, then $(t \rightharpoonup t^1) \cdot \sim (t \rightharpoonup t^k)$ is equivalent to $t = Df. t^1$, and t^1 is a term on condition. If some of t^i are t they are simply rejected as superfluous. If all t^i are t, then we have $(t \rightharpoonup t) \cdot \sim (t \rightharpoonup t^k)$. Since t is not a term before the acceptance of $t \rightharpoonup t$, the latter is not a definition. The matter of circularity is similar to I: if t^i contains a term defined through t, then t^i is not a term.

The requirement of adequacy for II^2 is guaranteed by point 3. The second and third requirements are met for II^2 in the sense that in $t^1, \ldots, t^m, t_1^1, \ldots, t_1^k, t_2^1, \ldots, t_2^l$ there is no term defined through t and not one of them is t.

4. DEFINITIONS AND ASSERTIONS

In the language of science definitions are formulated in literarily distinct forms: with the help of the expressions "is", "we will call", "if..., then we will call (name)", etc. These variations do not affect the essence of the definition: under all circumstances the definition is an agreement to designate some object by a term with a meaning.

Definitions are often formulated as sentences about objects rather than as agreements on terms. This is useful for inference. But it leads to confusion of logically different forms. All terms found in sentences about objects have a meaning independent of the sentences and prior to the construction thereof, while the terms newly introduced into a definition have meaning only by virtue of the definition. It will be correct to speak as follows: one obtains sentences from definitions according to certain rules (which we take up below). Imparting to definitions the form of sentences about objects one enunciates sentences obtainable from those definitions which, in such a case, remain unclear (implicit).

One should not confuse the definition with the establishment of whether or not a given object belongs to some set (i.e., can be named by a given term or not). For example, the expression "If litmus paper immersed in a liquid turns red the liquid is acid" can be seen as a definition of the term "acid" (felicitous or otherwise) or, more explicitly, "a liquid turning litmus paper red is (called) acid". But it can be taken as one of the ways of ascertaining whether or not a given liquid is acid; in such a case the term "acid" is defined before this sentence and independent of it.

Usually one talks about implications from definitions. This is an in-accurate use of words since definitions do not have truth-value (the predicates "true", "false", etc., are not applicable to them).

In fact there is a special type of rule, which is not explicitly formulated in logic and which makes it possible to draw conclusions not from the definitions themselves but from assertions to the effect that the definitions are accepted. The schema of such rules is:

'If *D* is accepted, then *X*, provided that *Y*", where *D* is the formulation of a definition, *X* is an assertion whose form depends on the form of *D*, *Y* is a condition of the truth of *X*. In such a case, if *D* is really accepted (i.e., the assertion "*D* is accepted" is true), then *X* will be true in function of the property of the form "If..., then..." (*Y* can be empty).

5. DEFINITION AND SELECTION

Every definition of terms is connected with some method of selection of objects. But the selection of objects, resulting in the introduction of terms, is not always a definition. Other methods of introducing terms are, for example: enumeration of objects, so chosen that they have a single property in common; uniqueness achieved by choosing examples according to circumstances (especially the character of the reader) in such a way that the number and types of examples can vary; the term introduced designates that which the objects in the examples have in common; the task then consists in bringing the reader or listener to select the requisite property of the objects. Even though a new term is introduced, this is not definition.

The case is the same when one finds operations which make it possible to discover or create an object and to introduce a term for it. The case is essentially the same but somewhat more complex: "the object which you see (hear, etc.) is called *t*". Although we here use terms with known values we are not establishing relations between the meanings of terms, as in cases I to IV.

When one talks of "operational definitions" one has in mind the construction of terms through description of the operations for the selection of objects, which is a mode of introducing terms, different from definition of terms in our sense. We assume that such methods are available although we will not examine them. The most that can be in general said about them (without going into the concrete operations) is the above remark.

More precision is needed on the expression "use of some terms to form others". In the cases with which we are dealing one uses language, i.e., one uses certain terms. But they are used as means for selecting objects and means for the orientation of the investigator in the world. Terms are not provided here as object of our attention and are not used as material for the formation of new terms. The terms introduced in such cases are simple in construction.

6. CONCEPT

D1. A term, the meaning of which has been established by definition (which is introduced, created by definition) will be called a concept.

It is clear that not every term will be a concept even if it is complex in construction. Thus, $\{\alpha; t^1, \ldots, t^n\}$ is not a concept. It is possible to find terms constructed out of concepts, but not being concepts. For example, the term "10 kg m/sec" is constructed from the concepts "number 10", "kg", "m" and "sec" but it is not a concept if it is the result of substituting a sentence with this term for sentences with the terms "*a* kg", "*b* m" and "*s* sec".

D2. In the instance of the definition $t^1 = Df.t^2$ the content of the concept of t^1 is the meaning of t^2; in the case of the definitions II to IV the content of the concept of t is the meaning of all the terms in t, which figure in the definition.

D3. Concepts appear in every known domain of science and can be divided into specific and non-specific. In turn the specific concepts can be divided into primitive (not defined through other specific concepts) and derived (defined through other specific concepts – ultimately through primitive ones). Primitive concepts are defined with the help of the terms of ordinary language, of the terms of other sciences and even of the concepts of other sciences. There is no such thing as an absolutely indefinable concept. The primitive concepts in axiomatic theories are "defined by the axioms" (explained below). Undefined (primitive) "signs" are used only in formal constructions. However, without interpretation they have no meaning, i.e., they are not signs (and, consequently, terms).

There are cases where terms are accepted simply because they are useful for defining terms which are necessary for one reason or another. Three outcomes are possible in such cases: 1) these terms are simple in construction; they serve to introduce the primitive concepts; 2) the meaning of these terms is explained with the help of other terms, so that this "explanation" contains an implicit definition; 3) in the "explanation", mentioned in 2), there is no definition; with the help of these terms one introduces primitive concepts (and the former are not such). In practice, elements of the three variants are mixed together and the introduction of logical clarity is not only extremely difficult but often useless.

D4. There is in use a term t^1, and a concept t^2 is introduced (defined) in such a way that $t^1 \rightleftharpoons t^2$ is the case. Such an introduction of t^2 is called an explication of t^1.

Practically it can often happen that the explicandum and explicans are identical in meaning. Usually the terms to be explicated are vague, ambiguous, etc. As a result of the explication we get concepts which coincide in meaning with the explicanda only in certain contexts. From this point of view explication serves not only to render precise the sense of terms (which can be done through definition) but also to select (distinguish) the domain of investigation. The investigator introduces the term t^2 in order to use it in place of t^1 and he makes the necessary decision (i.e., carries out a definite selection). This in itself eliminates the ambiguity of the terms.

7. MEANING AND CONTEXT

D1. The context of a term is some set of sentences and terms, localized in space or time, in which it occurs (where it is met, used, etc.).

The symbols

$$C^i \text{ and } tC^i$$

will represent suitable contexts and the term t in these contexts.

D2. If for any two different contexts C^1 and C^2 it is the case that

$$tC^1 \rightleftharpoons tC^2$$

then t is a non-contextual or unambiguous term (its meaning does not depend on context).

D3. If one finds at least two contexts C^1 and C^2 such that

$$\sim (tC^1 \rightleftharpoons tC^2)$$

then t is contextual or ambiguous (its meaning depends on context).

We exclude ambiguity of terms, agreeing: *A1.* If $\sim(tC^1 \equiv tC^2)$ then tC^1 and tC^2 are distinct terms.

8. THE REDUCTION PROBLEM

One of the problems of the logical analysis of scientific knowledge is that of the reduction of scientific terms of any type and of any degree of complexity to terms which are simple in construction. One can imagine two ways of solving the problem: 1) by indicating methods of replacing any scientific term with some ordered set of terms which are simple in construction; 2) by indicating methods of constructing scientific terms so as to explain all connections of terms as to their value, up to and including terms which are simple in construction. In the first sense the problem is unsolvable; in the second it is *de facto* solved in every domain of science. Of course, this solution is not something given once and for all. There are no absolute criteria for the simplicity and complexity, or for the primitive and derivative character of concepts. An important role here is played by historical conditions, chance and the constant change of science. The same results can often be obtained through different sets of terms, different ways of introducing them, and different systems of relating them to each other. Therefore, the principles of logic concerning the relationships of terms and the methods of constructing them are just schemata which aid in studying the terminology of some domain of science at a certain stage in its development (which brings improvements in the terminological apparatus of science), but are not at all absolutely binding in the establishment of relations like "simple – complex", "primitive – derived", etc.

Of course, every term of science is ultimately based on some set of terms which are simple in construction. But the "distance" between a given term and this base is often quite great (mediated) and the path itself is sometimes so convoluted that reduction to the base is of no practical import. The assertion that one can find such a "basic" equivalence for every term remains an abstract possibility. Attempts to do it on any large scale lead to failure. But this does not mean that it would be useless to try to do this for limited domains of science – fragmentarily, simplistically, approximately, etc. The practical and theoretical importance of such efforts seems to be growing.

9. TERMS OF TERMS

If an object is a term this means that it fills a certain function in the activity of the investigator; it plays a role and is used in a certain way. But a term can itself be the object of attention as is the case, for example, in logic. One needs a term for this case. The peculiarity of this situation is that the term for this term will be some example of the term itself, with some expansions or modifications (quotes, italics, etc.). For the term of the term t we will use

$$[t]$$

For t and $[t]$ it will be the case that

A1. $\sim(t \rightharpoonup [t])$, $\sim([t] \rightharpoonup t)$.

A2. The meaning of t does not depend on $[t]$ (i.e., it is known before the construction of $[t]$). But the meaning of $[t]$ is known if and only if the meaning of t is known.

If the meaning of t is not known then it is considered not as a term but as a perceptible object, having the form t, and then $[t]$ is not a term for a term but simply the term of some object.

CHAPTER FOUR

SENTENCES

1. THE PROBLEM OF DEFINING SENTENCES

There are definitions of sentences through truth-values. This is in particular possible if one assumes that a sentence is that which can be true or false. But this is an insufficient definition. One has to know what a sentence is before one talks about its properties such as truth, falsity, etc. While one can attain a certain illusion of immediate clarity in reference to truth and falsity, for other truth-values (which are possible and have to be included in a definition of truth) this is not possible. Relative to certain forms of sentences even the terms "true" and "false" have no immediate clarity.

There are also definitions of sentences with the help of terms like "thought", "content", "assertion", "negation", etc. In particular, a sentence is sometimes defined as a thought affirming or denying something about something. These definitions are also insufficient. Affirmation and denial are forms of sentences. The terms "thought", "content", etc. are not sufficiently defined; they are ambiguous. Using them in the present context is a leftover of psychologism in logic, which holds that the linguistic form in itself is of no interest for logic; it is merely the envelope of ideal (spiritual) objects existing somewhere in the head.

Sentences are empirically given (perceptible) objects, constructed from terms according to certain rules and with the help of supplemental, perceptible objects. These are special sorts of "entitative" structures, i.e., structures of perceptible "things". And the definition of the term "sentence" has to be found by describing these structures. The following have to be kept in mind. First, the number of different sentential structures is not limited by any circumstance flowing from the nature of the sentence itself. It is only because the introduction of new structures depends not on abstract possibilities but on the cognitive needs of people that the set of sentential structures is limited and relatively small. By its very nature the definition is limited. Second, sentences can be viewed from different angles. The structural descriptions from one point of view should be distinguished from those from another point of view. It can be said that the description of sentential structure is a process in many "dimensions". The definition sought after can be constructed only from the set of definitions provided in the different sections of logic.

We will take up the following way of defining the concept "sentence":

1) enumeration of the structures, made up of terms and operators, which are considered sentences;

2) enumeration of structures, made up of sentences and operators, which are considered sentences;

3) if Y is a sentence and there is an agreement to the effect that X is identical with Y as to meaning, then X is a sentence.

In the following paragraphs we will enumerate the structures referred to in points 1) and 2). The possibilities of introducing new structures are unlimited, according to 3). In this section we will take up some general questions, where the ordinary understanding of sentences will be enough and can be elucidated through examples.

2. BASIC PRINCIPLES OF THE CONSTRUCTION OF SENTENCES

Sentences will be represented by the symbols

$$X, Y, Z, X^1, X^2, \ldots, Y^1, Y^2, \ldots.$$

Each of these symbols is any sentence and a difference in symbols used means only a difference in sentences.

Every sentence is made up of terms and of some other objects (e.g., logical signs). Let $t^1, \ldots, t^n$ $(n \geqslant 2)$ be all the terms found in a sentence X. We make the following assumptions (corresponding to the factual state of affairs):

A1. The existence of objects $\Pi^1, \ldots, \Pi^n$, designated by the terms $t^1, \ldots, t^n$, does not depend on X (on whether or not X is constructed).

A2. The selection of objects $\Pi^1, \ldots, \Pi^n$ does not depend on X (they can be selected without X being constructed).

A3. The meaning of $t^1, \ldots, t^n$ does not depend on the construction of X (it is known before the construction of X).

Assuming *A1–A3* saves us from a whole series of consequences which are undesirable and unavoidable without them. Let us take, for example, the phrase "This sentence which I am now writing (reading, enunciating, etc.) is not true". This is not a sentence since the expression "This sentence I am now writing" is void of meaning as long as the sentence is not written; i.e., it depends on the phrase as a whole and we cannot select the corresponding object without writing the phrase.

It turns out that the structure of an expression is not sufficient to decide whether the requirements of *A1–A4* are met or not. These requirements are the non-formal conditions of the construction of sentences. And in each particular case it has to be clear whether we have to do with a sentence or a sentence-like object. There are no formal (structural) criteria for distinguishing them (as in the case of signs and sign-like objects).

A4. $(t^1 \rightleftharpoons t^2) \rightarrow (X \rightarrow X(t^1/t^2))$,
where $X(t^1/t^2)$ is a sentence formed from X by substituting t^2 for t^1.

3. SENTENTIAL OPERATORS

D1. Sentential operators are objects which permit the formation of sentences from terms and from sentences.

D2. Operators of terms and sentences are logical operators. Operators of terms and sentences are different operators. We will examine this difference more in detail below. Neither in ordinary language nor in the language of science are these operators always clearly and unambiguously expressed. However, we are obliged to assume that they are distinct, perceptible objects, localised in space and time. We also leave aside the fact of the ambiguity of linguistic expressions and the many ways of expressing one and the same function (filling the role) of signs. These abstractions mean the following: in real languages there is something which corresponds to the symbols used to designate the operators in question in the theory of knowledge; these symbols are unambiguous and their visible difference is an indicator of the different functions of the corresponding linguistic forms. In other words, we here abstract the function of the linguistic forms, whatever form they may take. On the other hand, these abstractions indicate the assumption of some devices necessary for distinguishing these functions in any context of the language in question.

The operators considered here are not signs in their own right. They become parts of signs only in conjunction with terms or sentences. Investigating them is not a matter of describing their perceptible form (which is taken as given) but of establishing their role in the formation of signs. The latter is not possible without a description of the properties of the structures, of which they are elements. But to describe these structures means to describe the conditions of the meaningfulness (*osmyslennost'*) of terms and of the truth-value of sentences, the rules of construction of terms and sentences from others, etc., i.e., to investigate knowledge on the planes of construction and meaning.

The study of some operators can lead to the study of others if it is possible to define the structures containing the former through those containing the latter. One has to distinguish the reducibility of logical operators from the use of some of them in the description of the properties of others. If the description of all the properties of one operator is impossible without the use of another, then the second will be called necessary in reference to the first. Investigation of reducibility and of such dependence of operators leads to certain complex, mutually irreducible and mutually necessary operators, the study of which forms the basis of the logical theory of scientific knowledge.

Logical operators can be used to define the other logical devices which (together with the operators) are called logical signs. Although this is not a fully correct use of words, we will accept it, noting that the word "logical" here needs some clarification: logical signs are signs, the meaning of which is investigated and established in logic.

4. COMPLEX TERMS AND SENTENCES

Sometimes sentences are considered as complex assertory terms. It is of course legitimate to regard terms and sentences from some general point of view: both are structures of signs. But, one must keep the following in mind. Let us take, for example, the sentence "The particle is positively charged" and the term "The particle which is positively charged" (or the term "The fact that the particle is positively charged"). It is asserted in the first that the object has some property. There is nothing like this in the second. But a term "is affirmed". What does this mean? One of two things: either the construction of a sentence "There is (given, observed, etc.) a particle which is positively charged", or the transition to the sentence "The particle is positively charged". But in the first case we get a sentence which is different from the one given and in the second we have not an example of the reduction of a sentence to a term but only an example of the assertion that one can get a term from a sentence (and vice versa) through certain operations. In other cases the irreducibility of sentences to terms is even more evident. Thus, transformation of the sentence "All even numbers are divisible by two" into the term "Even number which is divisible by two" leads either to loss of the quantifier "all" or to the formation of a term which does not correspond to the initial sentence: in both cases the limitation expressed by the word "which" loses its meaning. Sentences are distinguished from complex terms by the presence of special operators which differ from those used for terms. But this is not all. Let us take, for example, the term "*a*, which is bigger than *b* and has property *c*" and the sentence "*a*, which has the property *c*, is bigger than *b*". The presence of the expressions "which" and "has the property" in both does not hide their differences. It is clear that it is not only the presence of such logical operators that is important but also their order as well as that of the terms.

5. SIMPLE AND COMPLEX SENTENCES

D1. Sentences which do not contain (contain) as parts at least two different sentences will be called simple (complex) in structure. In other words, structurally complex sentences are formed by the conjunction of one or more sentences with special operators.

D2. We will say that a sentence *X* is contentfully used in the obtaining of sentence *Y* if we find in *Y* at least one term or at least one sentence found in *X*.

D3. A sentence, for the obtaining of which no other sentence is contentfully used, will be called simple in construction. A sentence, for the obtaining of which at least one other sentence, different from the first, is contentfully used, will be called complex in construction.

Sentences simple in structure result from observation (empirical sentences) or presuppositions (assumptions, hypotheses). Sentences which are simple in structure can be complex in construction.

6. COGNITIVE ACTIVITIES

The process of obtaining knowledge is composed of the activities of the investigator with some given materials – sense-data or perceived objects – and already had knowledge. These activities are formulated in language by expressions like "we take", "we assume", "from...we get...", "putting ...for...", etc. These expressions are signs of cognitive activity.

Activities with perceived objects can be reduced either to activities with real objects (types of activities coincide) or to activities with sentences of the type "Let there be...", "We take...", "We assume that...", etc., which can be reduced to activities with sentences "if..., then..." (where the assumption is put after "if").

Activities with sense-given objects assume sense-reflection of the primitive material and of the results of the activities. They are included in the selection of the objects, in the distinguishing and comparing of them, in dividing, uniting, transposing, modifying, etc. them, in their inclusion and exclusion relative to certain connections, etc. Some of these activities influence the object; others do not. The primitive material and the result of activities of the second order can be described in sentences and the activity of the investigator can be eliminated with the help of conditional sentences of the form "If X then Y", where X is the description of the result of an activity and Y is knowledge obtained in these circumstances. The result is that activity of the second order can be considered the contentful condition of the obtaining of knowledge, which activities of the first order are not. They are the conditions of any cognitive activities with objects at all.

Knowledge is basically the sense perception of objects and all that has been said applies to it. But when knowledge is taken as a special sort of perceptible object (term, sentence) there appears a cognitive activity which is not reducible to the content of sentences and which involves the purely formal conditions of the obtaining of knowledge. This is the activity of obtaining some terms and sentences from others.

The task relevant to cognitive activities is to find the elementary cognitive activities which go to make up all the complex activities used to obtain knowledge. As already indicated, the elementary cognitive activities with objects are selection and comparison. We will call the selection of objects in the construction (obtaining) of sentences heuristic; similarly for comparison (*D1*). Investigation of the activities pertaining to knowledge is the basic task of the logical theory of scientific knowledge.

7. THE CONSTRUCTION OF SENTENCES

When one talks about the construction of sentences, one is not talking simply about their physical formation (which poses no real problem). One has in mind the formation of sentences which serve some purpose, which are accepted, recognized, etc. One can introduce and write down any number of sentences. But by no means all of them will be able to be accepted and recognized as having value for people. The question, what are the criteria of the practical value of sentences for some branch of science, is not a logical question. Logic does have something to do with the study of the criteria which enable one to accept or reject sentences.

Similarly, when one talks about obtaining (constructing) sentences from others, one has in mind not the physical transformation of one into the other but the activity which can be extrinsically described as follows:

1) Sentences $X^1,\ldots, X^n$ $(n \geqslant 1)$ are given as visible, audible, etc., objects; they are analyzed as to the terms and logical signs they contain and how these are related to one another;

2) pursuant to this analysis and to some other extrinsic circumstances (conditions, purpose, etc.) one establishes (produces, etc.) a sentence Y; obviously there are rules which (in these circumstances and for these purposes) permit the establishment of Y;

3) these rules are such that the following condition is met: if one accepts (recognizes) $X^1,\ldots, X^n$, then Y, obtained in accordance with these rules, has to be accepted.

Once these rules are established and formed, things change: they make it possible to accept Y once $X^1,\ldots, X^n$ are accepted. Now they are experienced as something like a force and law of nature rather than as a product of human creativity.

The rules of logical entailment occupy a special place among the rules for the obtaining of sentences. They are elaborated in such a way that the following postulate be satisfied: if Y logically follows from X, and X is taken as true, then Y has to be taken as true. The rules of logical entailment define the properties of the logical signs found in X and Y.

Thus, if one obtains Y from X according to these rules, this is due to the properties of the logical signs they contain. The signs are of such a nature that Y follows or does not follow from X depending on the structures of X and Y. X is here sufficient for the obtaining and accepting of Y. The sentence X is called premiss and Y is conclusion (or consequence).

Different from the rules of logical entailment are the rules which make it possible to accept Y, when $X^1,\ldots, X^n$ are known, but which do not define the logical signs contained in $X^1,\ldots, X^n$, Y. They can be divided into two groups. One of them includes agreements to substitute certain sentences for others and the conclusions of such agreements, according to which from the terms figuring in $X^1,\ldots, X^n$ we obtain new terms figuring in Y. We call them rules of substitution of terms. The substitutions carried out in accordance with them are means suitable for the preservation of and operation with knowledge. In this way complex sets of sentences are replaced by abbreviations, accessible to review as sets of signs. This is not symbolization, which is also abbreviation, but the substitution of some sets of sentences for others.

For example, in the case of sentences about the velocity of a body the substitution will not be the replacement of the words "kilometer" and "second" by the abbreviations "km" and "sec" but the replacement of sentences about the distance covered by the moving body and the time in which this occurred, by one sentence. In such substitutions the logical properties of the sets of sentences are clearly formulated. What is more, it is often only such substitutions which make it possible to bring out the rules of inference used (especially the logical and mathematical rules). For example, putting $a=f(\alpha, t)$ for $a=b^1,\ldots, a=b^n$, where α is a constant, t is time, and f a certain type of function, we can use the property f in subsequent judgements. The rule of substitution generated here is a simple, one-act agreement to use a sentence Y instead of some set of sentences X.

The rules for the substitution of terms (like the rules of logical entailment) are purely formal, i.e., the transition which they permit is based on consideration of the perceptible form of the terms and sentences. These rules are called deductive. The process of elaborating (forming, selecting) these rules is not formal; it is a matter of creative operations which aim at the formation of rules such that they can be used formally (i.e., within certain limits). The number of such rules is not limited.

Among them we find rules used in all or some sciences, rules used once or over and over, etc. A large part of scientific activity involves such a formation of rules.

In the second group we find rules which are sometimes called non-formal, inductive or extra-logical. These rules make it possible to accept Y if $X^1, \ldots, X^n$ are true but offers no guarantee that Y will be true. There are cases where Y proves not to be true and has to be rejected. These rules are based on certain general assumptions on the objects being investigated. To some extent these rules can be studied by a logical theory of scientific knowledge although there is a tendency to exclude them from the domain of logic.

8. THE MEANING OF SENTENCES

Even simple examples show that there is no complete parallelism between the planes of value for terms and for sentences. Thus, the term "the prime number four" has value and meaning, while "four is a prime number" is false. However, the use of the terms "true", "false", etc., relevant to terms is based on a confusion.

D1. We will say that the investigator knows the meaning of a simple sentence if and only if he knows the meanings of all the component terms and logical signs, and he knows the meaning of a complex sentence if and only if he knows the meanings of all the sentences and logical signs making it up. A proposition has meaning if and only if its meaning is known to the investigator.

The expressions "the sentence has meaning" and "the sentence does not have meaning" are to be used only to the extent that one can construct objects resembling the sentences having meaning, and there are no structural characteristics which make it possible to distinguish them. The expression "the sentence does not have a meaning (is meaningless)" is sometimes used to indicate that it is not possible to establish that a sentence is true or false. We, however, will use other terms for this.

D2. The terms and simple sentences found in a given sentence can be called its units of meaning.

The identity of X and Y according to meaning will be written as

$$X \equiv Y.$$

The symbol

$$X(t^1/t^2)$$

will represent the sentence formed from X by substitution of t^2 for t^1. The symbol

$$X(t^1, \ldots, t^n/t_1, \ldots, t_n)$$

will be considered an abbreviation for

$$((X(t^1/t_1)) \ldots)(t^n/t_n).$$

D3. Two sentences are identical in meaning only through agreement or because of the following assertions:

A1. $(t^1 \equiv t^2) \rightarrow (X \equiv X(t^1/t^2))$
A2. $(X \equiv Y) \rightarrow (Z \equiv Z(X/Y))$,

where $Z(X/Y)$ is a sentence formed from Z through substitution of Y for X.

A3. $(X \equiv Y) \cdot (Y \equiv Z) \rightarrow (X \equiv Z)$.

9. DEFINITIONS WITH SENTENCES

The agreements mentioned in *D1*IV8 of the previous section are definitions of the type: "Let object X be a sentence such that $X \equiv Y$" (where Y is a sentence) or "In place of Y we will use X as identical to it in meaning". Such definitions introduce either new terms of new logical signs (lacking, of course, in Y). At the same time they are agreements to call certain objects, found in X, terms or logical signs. Obviously, object X has to be constructed in such a way that it has the structure of a sentence.

Instead of the expression "identical in meaning" for definitions which contain sentences, we often use the expressions "if...then..." and "if and only if".

10. TERMS FROM SENTENCES

From any sentence X one can form the term

$$\downarrow X,$$

which reads "The fact that X".

A1. If X is a sentence, then $\downarrow X$ is a term.

A2. $(X \equiv Y) \leftrightarrow (\downarrow X \rightleftharpoons \downarrow Y)$

A3. $(t^1 \rightleftharpoons t^2) \rightarrow \downarrow X \rightleftharpoons \downarrow X(t^1/t^2)$

The definitions discussed in the previous section are modifications of definitions of the type

$$\downarrow X = Df. \downarrow Y.$$

What is more, terms are introduced according to schema I:

$$t = Df. \downarrow X.$$

11. TRUTH-VALUES

Truth-values are the terms "true", "false", etc. They will be represented by

$$v^1, v^2, \ldots, v^n, v_1, v_2, \ldots.$$

One has to distinguish the establishment of the meaning of terms v_i and v^i and the explication of the truth-value of a given sentence. The latter cannot be done until the former is carried out. Below we will formulate the general principles affecting only the meaning of truth-values.

We assume the designations:

1) $X \leftarrow v^i$ means "X has the truth-value v^i", where v^i is one of v^1, v^2,... (e.g., "X is true", "X is false", etc.);

2) $\sim(X \leftarrow v^i)$ means "X does not have the truth-value v^i" ("X is not true", "X is not false", etc.);

3) $X \approx Y$ means "X and Y are equivalent";

4) $\sim(X \approx Y)$ means "X and Y are not equivalent";

D1. $X \approx Y$ if and only if for any v^i is met the requirement: every time that one of X and Y has the value v^i, the other will have exactly the same value v^i, i.e.,

$$(X \leftarrow v^i) \leftrightarrow (Y \leftarrow v^i).$$

T1. If $X \approx Y$ then for any v^i

$$\sim((X \leftarrow v^i) \cdot \sim (Y \leftarrow v^i)) \cdot \sim ((Y \leftarrow v^i) \cdot \sim (X \leftarrow v^i)).$$

Truth-values are introduced so as to meet the requirements:

A1. $(X \leftarrow v^i) : \sim (X \leftarrow v^i)$
A2. $(X \equiv Y) \rightarrow (X \approx Y)$.

A3. If $\sim (X \leftarrow v^i)$ then X has a truth-value other than v^i.

D2. The truth-values $v^1, \ldots, v^n$ $(n \geqslant 2)$ will be called basic if and only if for any pair v^i and v^k the following are fulfilled:

$$(X \leftarrow v^i) \rightarrow \sim (X \leftarrow v^k)$$
$$(X \leftarrow v^k) \rightarrow \sim (X \leftarrow v^i).$$

D3. The set of basic truth-values $v^1, \ldots, v^n$ will be considered complete if and only if

$$(X \leftarrow v^1) : \ldots : (X \leftarrow v^n)$$

D4. The truth-values $v_1, \ldots, v_m$ $(m \geqslant 1)$ will be called complementary to the basic truth-values if and only if for every v_j $(j = 1, \ldots, m)$ there is at least one basic truth-value v^i such that

$$(X \leftarrow v^i) \rightarrow (X \leftarrow v_j).$$

There is one privileged value found among v^i. This value is "true". For it we take the number 1, i.e., it will be represented as v^1. It has the following properties.

A4. The value "true" (v^1) is always found among the basic values.

A5. $(X \leftarrow v^1) \leftrightarrow X$

A6. The rest of the basic values are defined through v^1 according to

$$(X \leftarrow v^i) \equiv (Y \leftarrow v^1),$$

where v^i $(i = 2, \ldots, n)$ depends on the form of Y.

It is obvious that

T2. $((X \leftarrow v^i) \equiv (Y \leftarrow v^1)) \rightarrow ((X \leftarrow v^i) \leftrightarrow Y)$.

A7. Complementary truth-values are defined through the basic ones according to

$$(X \leftarrow v_j) \equiv (X \leftarrow v^i) : \ldots : (X \leftarrow v^k),$$

where $v^i, \ldots, v^k$ are values taken from among the basic ones and v_j is a complementary value.

Complementary values can also be introduced by means of definitions of the form

$$(v_j \rightharpoonup v^i)\cdot\ldots\cdot(v_j \rightharpoonup v^k)\cdot \sim(v_j \rightharpoonup v^l),$$

where v^l is any value different from $v^i,\ldots, v^k$.

T3. To the extent that all basic values are reduced to v^1, the complementary ones can also be reduced to v^1.

If the number of basic values is taken to be two and they are v^1 and v^2, then:

T4. $(X \leftarrow v^1) \rightarrow \sim (X \leftarrow v^2)$
T5. $(X \leftarrow v^2) \rightarrow \sim (X \leftarrow v^1)$

If, however, the number of basic values is greater than two, then:

T6. $\sim (X \leftarrow v^i) \rightarrow (X \leftarrow v^{k1}):\ldots:(X \leftarrow v^{km})$
$(X \leftarrow v^{k1}) \rightarrow \sim (X \leftarrow v^i),\ldots,(X \leftarrow v^{km}) \rightarrow \sim (X \leftarrow v^i),$

where $v^{k^1},\ldots, v^{k^m}$ are all the basic values, different from v^i.

The value "false" will be designated by the symbol v^n. It is clear that in general $\sim (X \leftarrow v^1)$ will not coincide with $X \leftarrow v^n$. So that

T7. $\sim (\sim (X \leftarrow v^1) \rightarrow (X \leftarrow v^n)),$
$\sim (\sim (X \leftarrow v^n) \rightarrow (X \leftarrow v^1)).$

For sentences which are themselves about truth-values the following assertion (according to *A6*) holds:

T8. $((X \leftarrow v^i) \leftarrow v^1) \leftrightarrow (X \leftarrow v^i)$
$(\sim (X \leftarrow v^i) \leftarrow v^1) \leftrightarrow \sim (X \leftarrow v^i).$

According to the theorems and definitions introduced above these sentences require only two basic truth-values.

We can therefore assume:

D5. $((X \leftarrow v^i) \leftarrow v^n) \equiv (\sim (X \leftarrow v^i) \leftarrow v^1).$

T9. If the number of basic values is greater than two, then

$$\sim ((X \leftarrow v^1) \approx X).$$

If $X \leftarrow v^2$ then when $n=3$ we get $(X \leftarrow v^1) \leftarrow v^3$. Whence the important conclusion:

T10. $\sim ((X \leftarrow v^1) \equiv X).$

Cases where it is impossible to establish the truth-values of sentences are excluded by the very mode of introducing truth-values. There are cases where it is not possible to say if a sentence is true or false. But this is some third value. For any value v^i it is the case that:

A8. If it is impossible to establish that $X \leftarrow v^i$, then $\sim (X \leftarrow v^i)$.

It is clear that the meaning and truth-value of a sentence are not one and the same thing. What is more, one can know the meaning of a sentence ("understand the sentence") and yet not know its truth-value or consider it not-true (true, etc.). If the investigator does not know the meaning of X then for him the sentence $X \leftarrow v^i$ *de facto* means the following: "Object X is a sentence for someone (for some other investigators) and $X \leftarrow v^i$". Such a situation is possible.

12. SENTENTIAL STRUCTURE AND TRUTH-VALUES

Definition of truth-values according to the schema introduced in the previous section presupposes the knowledge of the structures of sentences X and Y (Y is selected relevant to X and is not an arbitrarily selected sentence). Thus, the exact definitions of truth-values can be established only after one has fixed the structure of these sentences. Definitions have to be given for each structure. A full definition of each truth-value is possible only under the condition that all the sentential structures have been reviewed. And since there are no *a priori* limits here, it is impossible to have a final and complete definition of each truth-value. Invention of some new sentential structure requires special definitions of truth-values relative to this structure.

Definitions have to be given for each structure since definitions which are effective for one structure are not for another. A definition has to be constructed for a specific structure and not through a structure; this is not the same thing. If the sentential structure is given then we can sometimes ascertain (not introducing it by definition!) its truth-value if we know the truth-values of the component parts. For example, if we have $X \cdot Y$ we can ascertain its truth-value if we know those of X and Y. However, this is possible only when one has definitions of truth-values for this structure. And this means that the truth-value of $X \cdot Y$ here is determined through those of X and Y and is only a special case.

When certain sentential structures are reduced to others, then such a reduction is also possible for the truth-values, i.e., it is possible to define the truth values for certain structures through those for others. This makes possible assertions of the type

$$(Y^1 \leftarrow v^{i1})\cdot\ldots\cdot(Y^n \leftarrow v^{in}) \rightarrow (X \leftarrow v^k),$$

where $n \geqslant 1$. The sentence

$$(Y^1 \leftarrow v^{i1})\cdot\ldots\cdot(Y^n \leftarrow v^{in})$$

here constitutes the logical condition of the v^k value of X (*D1*). The special case is $n=1$, Y^1 is X and v^k is v^1.

The goal of the definitions we are discussing here is to enumerate for every truth-value v^k all the conditions of v^k value of sentences with a given structure.

13. THE NUMBER OF TRUTH-VALUES

The number of basic truth-values cannot be smaller than two but can be greater than two. Two-valued logic (the logical theory using only two basic truth-values) is only the simplest, special case. Two-valued logic is the most simple possible case if it takes the value v^1 and the value v^2 is $\sim (X \leftarrow v^1)$. Two-valued logic is a special case if one excludes cases where sentences can take other values (other than true and false) and only considers a limited set of true-false sentences.

We here assume that the number of sentential truth-values can be greater than two. In what follows we will use both a simple two-valued variant and a four-valued case. We will use the following basic values:

1) v^1 is true;
2) v^2 is indeterminate;
3) v^3 is unverifiable;
4) v^4 is false.

This set of basic values is complete. We will also use the following complementary values:

1) nv^1 is not-true;
2) nv^2 is determinate;
3) nv^3 is verifiable;

4) nv^4 is not-false;
5) nv^1nv^3 is verifiable-not-true;
6) nv^4nv^3 is verifiable-not-false;
7) nv^2nv^3 is verifiable-determinate.

Values 1 to 4 are here defined by the assertions:

D1. $(X \leftarrow nv^i) \equiv (X \leftarrow v^k):(X \leftarrow v^l):(X \leftarrow v^m)$,

where v^i, v^k, v^l and v^m are v^1, v^2, v^3, and v^4 in all possible permutations. Values 5 to 7 are defined:

D2. $(X \leftarrow nv^1nv^3) \equiv (X \leftarrow v^2):(X \leftarrow v^4)$
D3. $(X \leftarrow nv^4nv^3) \equiv (X \leftarrow v^1):(X \leftarrow v^2)$
D4. $(X \leftarrow nv^2nv^3) \equiv (X \leftarrow v^1):(X \leftarrow v^4)$.

That the number of truth-values can be greater than two is an empirically established fact. Suppose a given sentence is the result of observation and its verification is accomplished through comparison of it with some domain of observation. The sentence contains at least two units of meaning. And comparison of it with the elements (objects) of a given domain of observation is a procedure with at least two steps. Each step admits of a positive or negative result. There is thus the possibility of introducing at least four different terms for these results, each of which is a truth-value.

If the truth-value of a sentence can be established only by finding out whether it was obtained from other sentences through rules of logical inference or not, then at least the following cases are possible:

1) Y follows from X and then Y is considered true relative to X (Y is provable);

2) from X follows the negation of Y and then Y is considered not-true relative to X (Y is refutable);

3) if from X follows neither Y nor its negation, then Y is considered undecidable relative to X; if this happens for any X, then Y is undecidable in principle.

There is no need to go into the other cases. If the fact of the possibility of three and more results in the establishment of sentential truth-values is once established, then it has to be taken into account in the logical theory of scientific knowledge. The limitation to two values in classical logic means the consideration of only those cases where only two results are possible in the processes of the establishment of sentential truth-values (in the verification of sentences).

14. TRUTH

Therefore, all truth-values are defined through truth. The definition of the term "true" has to be given relative to the structure of sentences and means of obtaining them. There is no single definition of it, suitable for all cases of sentences.

Obviously, this will also be the case for other values defined through it. In general, one can only say the following.

The term "true" means above all that the investigator accepts the sentence, agrees with what is said in it, etc. This act of agreement and acceptance is some sort of primary, explicit operation which is not definable in logical terms.

But the investigator can have all sorts of reasons and motives for accepting a sentence (i.e., for accomplishing the act of agreement, assent, etc.): fear of punishment, stupidity, tactical considerations, etc. Of course, we have to exclude such cases and come up with a fearless, serious, unbiased and sufficiently knowledgeable investigator who will accept a sentence only under certain well-defined circumstances. Then the expression "*X* is true" (as distinguished from "*X* is considered true") will mean: the investigator accepts *X* only because certain of these circumstances are the case. To define the term "true" – and, therefore, the sentence "*X* is true" – means to enumerate these cases. And this can be done only if one constructs a logical theory of scientific knowledge, i.e., if one studies the concrete sentential structures, their interrelations and the methods of obtaining them (but not simply as a premiss to their construction). And it seems that we have to take into account the tradition which has been built up in science throughout history and which is passed on to our putative investigator.

There have been many cases where people have achieved what they set out to do by taking as true sentences which actually were false. They did certain things believing in the truth of these (false) sentences.

This opened the door to taking as true those sentences belief in the truth of which led to the desired (positive) effect in a given activity. We exclude such a pragmatic criterion for the evaluation of sentences and assume the following: the investigator accepts a sentence (considers it to be true) irrespective of whether or not he or someone else bases his conduct on it (is directed by it in his activities).

We might note in passing that people often come to grief by believing true sentences to be true; the denial of true sentences can also sometimes be useful. All of this is extra-logical.

It is evident that for sentences with different structure the term "true" is defined in a different way. For example, the defining portions of the expressions "$X \cdot Y$ is true" and "$X : Y$ is true" will contain different expressions (otherwise the signs "and" and "or" would not differ). But the term "true" can also have a different definition for sentences which have identical structures but which are obtained in different ways. For example, "If X, then Y" will be "true" in one way if it results from empirical investigation of objects, and in another if it follows from the logical connections of sentences (in particular the logical entailment of Y by X).

The illusion that the term "true" has one and the same meaning for sentences with one and the same (similar) structure is the result of a psychological transfer: the meaning of the term is known for one group of sentences and all other sentences with similar structures are considered from the same point of view. The illusion is possible because one limits consideration to the act of acceptance and forgets what it takes to accomplish this act.

In the case of sentences which are complex in construction the question of truth reduces to that of the truth of other sentences. For sentences which are simple in construction and which have been obtained through observation (sensation and perception) of some region of the world (of objects), the following hold:

1) a sentence obtained through observation (i.e., sensation and perception of objects) in some range of objects is true relative to that range of observation (errors of observation being excluded);

2) if some X is given and there is a range of observation, then X is true relative to this range if and only if it can be obtained therein; in other words, if X and the range of observation are given, then X is true relative to this range if and only if comparison of the elements of X and the elements of the range provides a result, corresponding to the definition of the term "true" for sentences with a structure similar to that of X.

15. VERIFICATION

D1. Verification of a sentence is establishment of its truth-value.

D2. Selection of objects during verification of a sentence is verificatory selection.

For verification of X it is necessary to have:

1) exact definitions of truth-values for sentences with analogous structures or obtained with analogous methods;

2) the possibility of constructing true $Y^1,\ldots, Y^n$ $(n \geqslant 1)$ such that $Y^1 \cdot \ldots \cdot Y^n \rightarrow (X \leftarrow v^i)$.

16. LOCAL AND UNIVERSAL SENTENCES

D1. Suppose t is found in X. If X has one truth-value in confrontation with one object, designated by t, and another in confrontation with another object t, then X is a t-local sentence (the truth-value of X depends on t). But, if X has one and the same truth-value in confrontation with any object t, then X is a t-universal sentence, (the truth-value of X does not depend on t). A sentence is local (universal) if and only if it contains (does not contain) a term, for which it is local (universal).

Let $X^1,\ldots, X^n$ $(n \geqslant 1)$ be all the sentences found in some assertion we make, the truth-values of which depend on t: we will assume that the truth-values of these sentences within the limits of the assertion in question will be established in each instance by confrontation with one and the same object, designated by t.

Let the truth of Y depend on that of X (If X is true, then Y is true), and let there be some assertion A which takes account of this dependence and which contains X and Y. There are four possible cases: 1) X and Y are both t-local; and the acceptance of A means that Y is true in confrontation with every object t, in confrontation with which X was true; 2) X is t-local and Y is universal; for the acceptance of the truth of Y it is enough that X be true in confrontation with any object t; 3) X is universal, and Y is t-local; if X is true, then Y has to be true in confrontation with any object t; 4) X and Y are both universal; no stipulations are needed here.

17. METASENTENCES

If sentences are regarded as special sorts of objects one can introduce terms to designate them. Normally one uses a sample of the sentence in question with some additions or physical modifications (brackets, italics, etc.). Let us represent the term designating the sentence X with the symbol

$$[X].$$

A sentence can be considered simply as an object of the type X or, more precisely, as a sentence about something. In the first case the relationship of X and $[X]$ is the simple relation of object and term; in the second we have the relation between the sentence and its name. We here consider the second.

For X and $[X]$ the following holds:

A1. The meaning of X does not depend on $[X]$ (is known before the formation of $[X]$). But, the meaning of $[X]$ is known if and only if that of X is known.

From this viewpoint the famous paradox of two "sentences", one of which asserts that the other is true while the other asserts that the first is false has an extremely trivial solution: these "sentences" are not really sentences since they do not satisfy *A1* and the above mentioned requirement (cf. Section 2) on the meaning of sentences in general.

D1. Sentences which contain terms of terms or terms of sentences are metasentences.

All logical assertions on terms and sentences are metasentences. Above we have not used brackets (or other signs) for reasons of simplicity, assuming that what was intended was clear from the context. The assertions made above can be more accurately written as follows:

$$[t^1] \rightharpoonup [t^2], \qquad [X] \equiv [Y], \quad \text{etc.}$$

But there is no need for such complications: with the exception of examples, we will not use extra-logical terms or assertions so that there will be no chance of confusing terms and sentences with metaterms and metasentences.

CHAPTER FIVE

SENTENTIAL LOGIC

1. SENTENTIAL LOGIC

The section of logic dealing with the properties of sentences with the signs "and", "or", "not" and others derived from them is called sentential logic. More precisely, logic of sentences considers only those sentences which have the structure

$$\sim X, \quad X \cdot Y, \quad X : Y, \quad X \vee Y, \quad X \supset Y$$

etc., where X, Y,... are any sentences.

Those sentences which have another structure (i.e., do not decompose into sentences and logical signs $\sim$, $\cdot$, $:$, $\vee$, $\subset$, ...) are here taken as elementary. They will be represented by

$$p, q, r, p^1, p^2, \ldots, q^1, q^2, \ldots .$$

Their properties are studied in other sections of logic.

D1. We assume the following definition of sentences:

1) an elementary sentence is a sentence;

2) if X is a sentence, then $\sim X$ is a sentence;

3) if $X^1, X^2, \ldots, X^n$ are sentences, then $X^1 \cdot X^2$, $X^1 \cdot X^2 \cdot \ldots \cdot X^n$, $X^1 : X^2$, $X^1 : X^2 : \ldots : X^n$ are sentences;

4) if Y is a sentence and by convention $X \equiv Y$, then X is a sentence;

5) something is a sentence only in virtue of points 1 through 4.

Up to now we have used the logical signs $\sim$, $\cdot$, $:$, $\vee$ as primitively clear. We now have to explicate them, i.e., to provide an exact description of the properties of sentences containing these signs.

2. THE MEANING OF SENTENCES

From the point of view of the meaning of sentences the task of explication of the logical signs reduces to showing how to reduce all possible sentential structures with these signs to some basic or canonic structure (form), the meaning of which is considered clear from other sources. We here take the following (*D1*) as basic forms:

1) elementary sentences, i.e.,

$$p, q, r, p^1, p^2, \dots;$$

2) their extrinsic negations, i.e.,

$$\sim p, \sim q, \sim r, \sim p^1, \sim p^2, \dots;$$

3) sentences of the type

$$X \cdot Y, \quad X : Y, \quad X^1 \cdot X^2 \cdot \dots \cdot X^n, \quad X^1 : X^2 : \dots : X^n,$$

where X, Y, $X^1, \dots, X^n$ are elementary sentences or their general negations.

The meaning of the other structures is defined through the basic ones with the help of the following assertions:

A1. $\sim\sim X \equiv X$

A2. $\sim(X \cdot Y) \equiv (X \cdot \sim Y) : (\sim X \cdot Y) : (\sim X \cdot \sim Y)$
$\sim(Y^1 \cdot \dots \cdot Y^n) \equiv X^1 : \dots : X^k,$

where $X^1, \dots, X^k$ are all possible sentences different from $(Y^1 \cdot \dots \cdot Y^n)$ only by the presence of $\sim$ before at least one of $Y^1, \dots, Y^n$.

A3. $\sim(X : Y) \equiv (X \cdot Y) : (\sim X \cdot \sim Y)$
$\sim(X^1 : \dots : X^n) \equiv (X^1 \cdot \dots \cdot X^n) : Y^1 : \dots : Y^l,$

where $Y^1, \dots, Y^1$ are all possible sentences different from $(X^1 \cdot \dots \cdot X^n)$ only by the presence of $\sim$ before all $X^1, \dots, X^n$ or before i $(1 \leqslant i \leqslant n-2)$ of them.

A4. $(X : Y) \cdot Z \equiv (X \cdot Z) : (Y \cdot Z)$
$(X^1 : \dots : X^n) \cdot (Y^1 : \dots : Y^m) \equiv (X^1 \cdot Y^1) : \dots :$
$: (X^1 \cdot Y^m) : \dots : (X^n \cdot Y^1) : \dots : (X^n \cdot Y^m)$
A5. $X \cdot (Y \cdot Z) \equiv (X \cdot Y \cdot Z)$
$X \cdot (Y^1 \cdot \dots \cdot Y^n) \equiv (X \cdot Y^1 \cdot \dots \cdot Y^n)$
A6. $X : (Y : Z) \equiv (X \cdot Y \cdot Z) : (X \cdot \sim Y \cdot \sim Z) :$
$: (\sim X \cdot Y \cdot \sim Z) : (\sim X \cdot \sim Y \cdot Z)$
$Z : (X^1 : \dots : X^n) \equiv (Z \cdot X^1 \cdot \dots \cdot X^n) : (Z \cdot Y^1) : \dots :$
$: (Z \cdot Y^l) : (\sim Z \cdot Y_1) : \dots : (\sim Z \cdot Y_s)$

where $Y^1, \ldots, Y^1$ are the same as in *A3* and $Y_1, \ldots, Y_s$ are all possible sentences, differing from $(X^1 \cdot \ldots \cdot X^n)$ only by the presence of $\sim$ before all $X^1, \ldots, X^n$, except one;

$$
\begin{aligned}
&(X_1 : \ldots : X_m):(X^1 : \ldots : X^n) \equiv (Z^1 \cdot X^1 \cdot \ldots \cdot X^n):(Z^1 \cdot Y^1): \ldots :\\
&:(Z^1 \cdot Y^l): \ldots :(Z^r \cdot X^1 \cdot \ldots \cdot X^n):(Z^r \cdot Y^1): \ldots :(Z^r \cdot Y^l):\\
&:(X_1 \cdot \ldots \cdot X_m \cdot Y_1): \ldots :(X_1 \cdot \ldots \cdot X_m \cdot Y_s):(V^1 \cdot Y_1): \ldots :(V^1 \cdot Y_s): \ldots :\\
&:(V^t \cdot Y_1): \ldots :(V^t \cdot Y_s),
\end{aligned}
$$

where $Z^1, \ldots, Z^r$ are all possible sentences, differing from $(X_1 \cdot \ldots \cdot X_m)$ only by the presence of $\sim$ before all $X_1, \ldots, X_m$, except one; $Y^1, \ldots, Y^l$, $Y_1, \ldots, Y_s$ are the same as above, $V^1, \ldots, V^t$ are all possible sentences, differing from $(X_1 \cdot \ldots \cdot X_m)$ only by the presence of $\sim$ before all $X_1, \ldots, X_m$ or before i $(1 \leqslant i \leqslant m-2)$ of them.

A7. $(X \vee Y) \equiv \sim(\sim X \cdot \sim Y)$
$(X^1 \vee X^2 \vee \ldots \vee X^n) \equiv \sim(\sim X^1 \cdot \sim X^2 \cdot \ldots \cdot \sim X^n)$
A8. $(X | Y) \equiv \sim(X \cdot Y)$
$(X^1 | X^2 | \ldots | X^n) \equiv \sim(X^1 \cdot X^2 \cdot \ldots \cdot X^n)$
A9. $(X \supset Y) \equiv \sim(X \cdot \sim Y)$
A10. $(X \supset\subset Y) \equiv (X \supset Y) \cdot (Y \supset X)$.

Now, using *A2*IV7 "If $X \equiv Y$, then $Z \equiv Z(X/Y)$" for any sentence W, constructed from elementary sentences and the signs $\sim$, $\cdot$, $:$, $\vee$, $|$, $\supset$, $\supset\subset$ (one can introduce other derivative signs), can be found a sentence V in basic form such that

$$W \equiv V.$$

In such a case we will say *(D2)* that W is presented in the basic form V (or is reducible to it).

The relations $\vee$ and $:$ are defined by the assertions:

T1. $X \vee Y \equiv X \cdot Y : \sim X \cdot Y : X \cdot \sim Y$
$X^1 \vee X^2 \vee \ldots \vee X^n \equiv X^1 \cdot X^2 \cdot \ldots \cdot X^n : Y^1 : \ldots : Y^k$,

where $Y^1, \ldots, Y^k$ are all sentences different from $X^1 \cdot X^2 \cdot \ldots \cdot X^n$ only in that before at least one of $X^1, X^2, \ldots, X^n$ there is one and only one $\sim$ and this sign is lacking before at least one of them.

T2. $X : Y \equiv X \cdot \sim Y \vee \sim X \cdot Y$
$X^1 : X^2 : \ldots : X^n \equiv X^1 \cdot \sim X^2 \cdot \ldots \cdot \sim X^n \vee X^2 \cdot \sim X^1 \cdot \ldots \cdot \sim$
$\sim X^n \vee \ldots \vee X^n \cdot \sim X^1 \cdot \ldots \cdot \sim X^{n-1}$

3. TRUTH-VALUES

Truth-values are assumed known for elementary sentences. For other structures they are established in accord with the definitions which we now formulate.

To begin with, we introduce two-valued definitions. The basic values are v^1 and nv^1. The latter is defined by

$$([X] \leftarrow nv^1) \equiv \sim ([X] \leftarrow v^1).$$

The set of these values is complete, i.e.,

$$([X] \leftarrow v^1):([X] \leftarrow nv^1).$$

In the general case nv^1 does not coincide with v^4.

For the structures under study the truth-values are defined by

A1. $([\sim X] \leftarrow v^1) \equiv ([X] \leftarrow nv^1)$

A2. $([X^1 \cdot \ldots \cdot X^n] \leftarrow v^1) \equiv ([X^1] \leftarrow v^1) \cdot \ldots \cdot ([X^n] \leftarrow v^1)$

A3. $([X^1 : \ldots : X^n] \leftarrow v^1) \equiv ([X^1 \cdot \sim X^2 \cdot \ldots \cdot \sim X^n] \leftarrow v^1):$
$:([X^2 \cdot \sim X^1 \cdot \ldots \cdot \sim X^n] \leftarrow v^1): \ldots :([X^n \cdot \sim X^1 \cdot \ldots \cdot \sim X^{n-1}] \leftarrow v^1).$

From these assertions we get assertions determining the truth-values for the rest of the structures.

T1. $([X^1 \cdot \ldots \cdot X^n] \leftarrow nv^1) \equiv ([Y^1] \leftarrow v^1): \ldots :([Y^k] \leftarrow v^1),$

where $Y^1, \ldots, Y^k$ are all possible sentences, differing from $X^1, \ldots, X^n$ only by the presence of $\sim$ before at least one of $X^1, \ldots, X^n$.

T2. $([X : Y] \leftarrow nv^1) \equiv ([X \cdot Y] \leftarrow v^1):([\sim X \cdot \sim Y] \leftarrow v^1)$
$([X^1 : \ldots : X^n] \leftarrow nv^1) \equiv ([Y^1] \leftarrow v^1): \ldots :([Y^l] \leftarrow v^1),$

where $Y^1, \ldots, Y^l$ are sentences among which are found $X^1 \cdot \ldots \cdot X^n$ and all possible sentences, differing from it by the presence of $\sim$ (one and only one) before all or before $i (1 \leqslant i \leqslant n-2)$ of $X^1, \ldots, X^n$.

T3. $([\sim X] \leftarrow nv^1) \equiv ([X] \leftarrow v^1).$

We note that the signs $\sim$, $\cdot$, and $:$ are not introduced here by *A1–A3*. Their meaning is known prior to these assertions. The latter are definitions of the expressions

$$[\sim X] \leftarrow v^1, [X^1 \cdot \ldots \cdot X^n] \leftarrow v^1, [X^1 : \ldots : X^n] \leftarrow v^1.$$

The meaning of the expression

$$[X] \leftarrow v^1$$

is assumed to be known: it ultimately reduces to the meanings of the expressions

$$[p] \leftarrow v^1 \quad \text{and} \quad [\sim p] \leftarrow nv^1 .$$

And the question as to how one establishes the truth-values of elementary sentences is assumed to be answered.

We will now construct definitions of truth-values for the sentential structures under study from the point of view that there are four possible truth-values, v^1, v^2, v^3, v^4. Now, of course, nv^1 will no longer be a basic value.

A4. $([\sim X] \leftarrow v^4) \equiv ([X] \leftarrow v^1)$
A5. $([\sim X] \leftarrow v^1) \equiv ([X] \leftarrow v^2) : ([X] \leftarrow v^3) : ([X] \leftarrow v^4)$.

Thus, X will not always be equivalent to $\sim\sim X$, i.e.

T4. $\sim(X \approx \sim\sim X)$.

On the other hand

T5. $\sim X \approx \sim\sim\sim X$

The following will also be true:

T6. $([X] \leftarrow v^1) \rightarrow ([\sim\sim X] \leftarrow v^1)$
$([\sim\sim X] \leftarrow v^1) \rightarrow ([X] \leftarrow v^1)$
$([X] \leftarrow nv^1) \rightarrow ([\sim\sim X] \leftarrow nv^1)$
$([\sim\sim X] \leftarrow nv^1) \rightarrow ([X] \leftarrow nv^1)$
A6. $([X^1 \cdot \ldots \cdot X^n] \leftarrow v^1) \equiv ([X^1] \leftarrow v^1) \cdot \ldots \cdot ([X^n] \leftarrow v^1)$.
A7. $([X^1 \cdot \ldots \cdot X^n] \leftarrow v^2) \equiv Y_1^1 : \ldots : Y_1^k$,

where $Y_1^1, \ldots, Y_1^k$ is a list of all the possible different cases where at least one of $X^1, \ldots, X^n$ has the value v^2 and all the rest have the value v^1; more precisely, $Y_1^1, \ldots, Y_1^k$ are all possible sentences $([X^1] \leftarrow v_1) \cdot \ldots \cdot ([X^n] \leftarrow v_n)$, where at least one of $v_1, \ldots, v_n$ is v^2 and the rest v^1. Assertion *A7* can be written in the form

$$([X^1 \cdot \ldots \cdot X^n] \leftarrow v^2) \equiv (([X^1] \leftarrow v^2) \vee \ldots \vee ([X^n] \leftarrow v^2)) \cdot$$
$$\cdot ([X^1] \leftarrow nv^3 nv^4) \cdot \ldots \cdot ([X^n] \leftarrow nv^3 nv^4).$$

A8. $([X^1 \cdot \ldots \cdot X^n] \leftarrow v^3) \equiv Y_2^1 : \ldots : Y_2^l$, where $Y_2^1, \ldots, Y_2^l$ are all possible sentences of the type $\cdot([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $\cdot([X^n] \leftarrow v_n)$, where at least one of $v_1, \ldots, v_n$ is v^3 and all the rest are v^1 or v^2 (in all possible combinations). Assertion *A8* can be written in the form

$$([X^1 \cdot \ldots \cdot X^n] \leftarrow v^3) \equiv (([X^1] \leftarrow v^3) \vee \ldots \vee ([X^n] \leftarrow v^3)) \cdot$$
$$\cdot([X^1] \leftarrow nv^4) \cdot \ldots \cdot ([X^n] \leftarrow nv^4)$$

A9. $([X^1 \cdot \ldots \cdot X^n] \leftarrow v^4) \equiv Y_3^1 : \ldots : Y_3^m,$

where $Y_3^1, \ldots, Y_3^m$ are all possible sentences of the type $([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $\cdot([X^n] \leftarrow v_n)$, where at least one of $v_1, \ldots, v_n$ is v^4 and all the rest are v^1, v^2 or v^3 (in all possible combinations) i.e.,

$$([X^1 \cdot \ldots \cdot X^n] \leftarrow v^4) \equiv ([X^1] \leftarrow v^4) \vee \ldots \vee ([X^n] \leftarrow v^4).$$

A10. $([X^1 : \ldots : X^n] \leftarrow v^1) \equiv Z_1^1 : \ldots : Z_1^k,$

where $Z_1^1, \ldots, Z_k^1$ are all possible sentences of the type $([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $\cdot([X^n] \leftarrow v_n)$, where one and only one of $v_1, \ldots, v_n$ is v^1 and all the rest are v^2, v^3 or v^4 (in all possible combinations).

A11. $([X^1 : \ldots : X^n] \leftarrow v^2) \equiv Z_2^1 : \ldots : Z_2^l,$

where $Z_2^1, \ldots, Z_2^l$ are all possible sentences of the type $([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $\cdot([X^n] \leftarrow v_n)$, where at least one of $v_1, \ldots, v_n$ is v^2 and the rest are v^4.

A12. $([X^1 : \ldots : X^n] \leftarrow v^3) \equiv Z_3^1 : \ldots : Z_3^m,$

where $Z_3^1, \ldots, Z_3^m$ are all possible sentences of the type $([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $([X^n] \leftarrow v_n)$, where at least one of $v_1, \ldots, v_n$ is v^3 and the rest are v^2 or v^4 (in all possible combinations).

A13. $([X^1 : \ldots : X^n] \leftarrow v^4) \equiv Z_4^1 : \ldots : Z_4^r,$

where $Z_4^1, \ldots, Z_4^r$ are all possible sentences of the type $([X^1] \leftarrow v_1) \cdot \ldots \cdot$ $\cdot([X^n] \leftarrow v_n)$, where all $v_1, \ldots, v_n$ are v^4 or at least two of $v_1, \ldots, v_n$ are v^1 and the rest v^2, v^3 or v^4 (in all possible combinations).

4. LOCAL AND UNIVERSAL SENTENCES

Let us take the sentence $X : Y$. If X and Y are local sentences, then to ascertain the truth of $X : Y$ it is necessary to ascertain the following:

1) that $([X\cdot\sim Y]\leftarrow v^1)$ or $([\sim X\cdot Y]\leftarrow v^1)$ or both are possible;
2) that $([X\cdot Y]\leftarrow v^1)$ and $([\sim X\cdot\sim Y]\leftarrow v^1)$ are impossible.

Thus if we know that $([X\cdot\sim Y]\leftarrow v^1)$ this is not yet enough to allow us to accept $([X:Y]\leftarrow v^1)$. If X and Y are universal, then for the truth of $X:Y$ it is sufficient and necessary that $X\cdot\sim Y$ or $\sim X\cdot Y$ be true. It is sufficient because if $[X\cdot\sim Y]\leftarrow v^1$ or $[\sim X\cdot Y]\leftarrow v^1$, then all other possibilities are excluded. Thus we get here: if $[X\cdot\sim Y]\leftarrow v^1$, then $[X:Y]\leftarrow v^1$; if $[\sim X\cdot Y]\leftarrow v^1$, then $[X:Y]\leftarrow v^1$.

The same is true in the case of other sentences and other truth-values, when for the establishment of truth-values one selects two or more possibilities out of those present in the definition.

The definitions refer to the general case. But limitations of a particular order are also possible; e.g., a recognition that elementary sentences are universal, the exclusion of indeterminacy, etc. Such definitions as introduced in logic courses presuppose implicitly or explicitly that elementary sentences are universal. And the definitions therefore have a formulation which differs somewhat from that used above. According to these definitions, if one knows the truth-values of all $X^1,\dots,X^n$, then those of $X^1\cdot\dots\cdot X^n$ and $X^1:\dots:X^n$ are also known.

The following additional postulates are necessary to the definitions for the case of universal elementary sentences:

A1. $([X\cdot\sim Y]\leftarrow v^1)\rightarrow([X:Y]\leftarrow v^1)$
$([\sim X\cdot Y]\leftarrow v^1)\rightarrow([X:Y]\leftarrow v^1)$
$([X^1\sim X^2\cdot\dots\cdot\sim X^n]\leftarrow v^1)\rightarrow([X^1:\dots:X^n]\leftarrow v^1),\dots,$
$([X^n\cdot\sim X^1\cdot\dots\cdot\sim X^{n-1}]\leftarrow v^1)\rightarrow([X^1:\dots:X^n]\leftarrow v^1)$

A2. $([X]\leftarrow v^4)\rightarrow([\sim X]\leftarrow v^1)$
$([X]\leftarrow v^3)\rightarrow([\sim X]\leftarrow v^1)$
$([X]\leftarrow v^2)\rightarrow([\sim X]\leftarrow v^1)$

A3. $Y_1^1\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^2),\dots,$
$Y_1^k\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^2)$

A4. $Y_2^1\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^3),\dots,$
$Y_2^l\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^3)$

A5. $Y_3^1\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^4),\dots,$
$Y_3^m\rightarrow([X^1\cdot\dots\cdot X^n]\leftarrow v^4)$

A6. $Z_1^1\rightarrow([X^1:\dots:X^n]\leftarrow v^1),\dots,$
$Z_1^k\rightarrow([X^1:\dots:X^n]\leftarrow v^1)$

A7. $Z_2^1 \rightarrow ([X^1:\ldots:X^n] \leftarrow v^2), \ldots,$
$Z_2^l \rightarrow ([X^1:\ldots:X^n] \leftarrow v^2)$
A8. $Z_3^1 \rightarrow ([X^1:\ldots:X^n] \leftarrow v^3), \ldots,$
$Z_3^m \rightarrow ([X^1:\ldots:X^n] \leftarrow v^3)$
A9. $Z_4^1 \rightarrow ([X^1:\ldots:X^n] \leftarrow v^4), \ldots,$
$Z_4^r \rightarrow ([X^1:\ldots:X^n] \leftarrow v^4)$

5. TYPES OF SENTENCES

D1. If $[X] \leftarrow v^1$ for any combination of truth-values of the elementary sentences found in X, then X is called always-true or tautologous. In other words, if $p^1, \ldots, p^n$ are all possible elementary sentences found in X and if it is the case that

$$([p^1] \leftarrow v_1) \cdot \ldots \cdot ([p^n] \leftarrow v_n) \rightarrow ([X] \leftarrow v^1),$$

where $v_1, \ldots, v_n$ are the basic truth-values, pairwise different or identical in all combination, then X is a tautology.

D2. If $[X] \leftarrow nv^1$ for any combination of truth-values of elementary sentences occurring in X, then X is called an unsatisfiable sentence.

D3. If $[X] \leftarrow v^4$ for any combination of truth-values of elementary sentences occurring in X then X is called a contradiction.

T1. If X is a tautology (contradiction), then $\sim X$ is a contradiction (tautology).

D4. If $[X] \leftarrow v^1$ for at least one combination of truth-values of elementary sentences occurring in X then X is called a satisfiable sentence.

A satisfiable sentence which is not a tautology is called empirical.

T2. If X is a tautology, then X is satisfiable.

What (from the viewpoint of *D1–D4*) a given X is, is explained according to the definitions of the previous sections. In particular

$$X : \sim X, \ \sim(X \cdot \sim X), \ \sim(X \cdot \sim X \cdot Y)$$

are tautologies and

$$\sim(X : \sim X), X \cdot \sim X, X \cdot \sim X \cdot Y$$

are contradictions. But the class of tautologies (and of contradictions) can be defined through an axiomatic system. In such an axiomatic definition one selects certain tautologies as basic (axioms) and indicates rules which serve for obtaining from them any other tautology (rules of inference from the axioms). Classical sentential calculus as presented in textbooks is such an axiomatic construction embracing all two-valued tautologies on the level of sentential logic.

The difference between contradiction and unsatisfiability lies in the fact that in the case of a four-valued characterization of sentences there are cases where $[X] \leftarrow nv^1$ and $[X] \leftarrow nv^4$.

We assume some sentences to be true for reasons other than those mentioned in *D1*.

6. TRUTH FUNCTIONS

It is necessary to distinguish definitions of truth-values of structures already given from the definition (introduction) of new structures through truth-values. In the first case the value of the logical signs found in the structures is known before the definition; in the second it is established only through the definition. In the first case the sense of $\sim$, $\cdot$, $:$, ... is known prior to the definition and they are therefore found in definiens and definiendum. In the second case the definitions have the following form: we use the symbol $F(X^1,..., X^n)$ $(n \geqslant 1)$ to designate a sentence constructed from $X^1,..., X^n$ and having the following properties: if $([X^1] \leftarrow v_{11}) \cdot ... \cdot ([X^n] \leftarrow v_{n1})$, then this sentence has the value v^{i1}; if $([X^1] \leftarrow v_{12}) \cdot ... \cdot ([X^n] \leftarrow v_{n2})$, then it has the value v^{i2};...; if $([X^1] \leftarrow v_{1m}) \cdot ... \cdot ([X^n] \leftarrow v_{nm})$, then it has the value v^{im} (here $m = 2^{2^n}$, v^{ij} and v_{ij} can be identical and different in all possible combinations. Such a way of introducing F is called functional (by means of a matrix).

In introducing the sign F (in the second case) we have to carry out its interpretation right away, i.e., we have to find cases where one obtains a sentence constructed from $X^1,..., X^n$ and assuming the values $v^{i1},..., v^{im}$ for the corresponding combination of values $v_{1i},..., v_{ni}$ of sentences $X^1,..., X^n$. And the verification of this sentence has to be effected independently of the sign F. The latter can be introduced if the verification satisfies the condition indicated in the definition of F. Carrying out this process to the limit we arrive at a set of sentences with the signs $\cdot$, $:$ and $\sim$. So that the latter cannot be introduced into logic of science by means of definitions of the second type. They themselves are the condition thereof. It is interesting to note that if we introduce them by means of definitions of the second type then in the interpretation of F we will have to refer to them. For example, let us construct the definition: if a sentence constructed from X and Y has the value v^1 for $[X] \leftarrow v^1$ and $[Y] \leftarrow v^1$, value v^4 for $([X] \leftarrow nv^3) \cdot ([Y] \leftarrow nv^3) \cdot (([X] \leftarrow v^4) \vee ([Y] \leftarrow v^4))$, value v^2 for $([X] \leftarrow nv^3) \cdot ([Y] \leftarrow nv^3) \cdot (([X] \leftarrow v^2) \vee ([Y] \leftarrow v^2))$ and value v^3 for $([X] \leftarrow v^3) \vee ([Y] \leftarrow v^3)$, then we will represent it as $X \& Y$. Now, when will we use this $X \& Y$? Obviously, when $X \cdot Y$, where the value of $\cdot$ does not depend on the definition in question. But then the & is superfluous and the definition has to be constructed as was the case at the beginning of this section.

In short, whatever the signs we want to introduce through truth-value definitions, they ultimately have to be interpreted with the help of the signs $\cdot$, : and $\sim$. So that if $F^*(X^1, \ldots, X^n)$ is a sentence constructed from $X^1, \ldots, X^n$ with the help of $\cdot$, :, $\sim$ and possibly other signs, then for any $F(X^1, \ldots, X^n)$ there is an $F^*(X^1, \ldots, X^n)$ such that its truth-value definition coincides with the definition of $F(X^1, \ldots, X^n)$.

7. TRUTH CONDITIONS

Let X be a sentence of the type examined in this chapter. Let v^1 and nv^1 form a complete set of basic truth-values.

D1. We take the assertion

$$([p^1] \leftarrow \alpha^1) \cdot \ldots \cdot ([p^n] \leftarrow \alpha^n) \rightarrow ([X] \leftarrow v^1),$$

where $p^1, \ldots, p^n$ are elementary sentences, among which are all elementary sentences (under the above condition) found in X, and $\alpha^1, \ldots, \alpha^n$ is any combination of v^1 and nv^1. Let this assertion be true by virtue of *A1*V3–*A3*V3. In this case we will call

$$([p^1] \leftarrow \alpha^1) \cdot \ldots \cdot ([p^n] \leftarrow \alpha^n)$$

a strong logical truth condition of X.

D2. The definition of a weak logical truth condition of X differs from *D1* only through the admission that $p^1, \ldots, p^n$ contains at least one elementary sentence (not necessarily all) found in X.

If every strong logical truth condition of X is also that of Y, we will write

$$X \mathrel{\#\!\!\!\!\triangleright} Y.$$

An analogous relation for weak logical truth condition of X and Y will be written as

$$X \not\triangleright Y.$$

T1. If in X and Y there are no identical elementary sentences then $\sim(X \not\triangleright Y)$.

T2. $\sim(\sim pp \not\triangleright q), \sim(p \not\triangleright q \vee \sim q), \sim(p \not\not\triangleright p \vee q)$

T3. If Y contains at least one elementary sentence which is lacking in X, then $\sim(X \not\not\triangleright Y)$.

T4. $(X \not\not\triangleright Y) \rightarrow (X \not\triangleright Y)$
T5. $(X \not\triangleright Y) \cdot (Y \not\triangleright Z) \rightarrow (X \not\triangleright Z)$

T6. If $X \not\triangleright Y$, then $X \supset Y$ is a tautology; if $X \supset Y$ is a tautology and in X and Y there is at least one identical elementary sentence, then $X \not\triangleright Y$; if $X \supset Y$ is a tautology and Y contains no elementary sentences which are lacking in X, then $X \not\not\triangleright Y$.

8. THE CONSTRUCTION OF SENTENCES

If $X^1, \ldots, X^n$ are universal sentences, then for the construction of a true $X^1 \cdot \ldots \cdot X^n$ it is sufficient that all $X^1, \ldots, X^n$ be true and for the construction of a true $X^1 : \ldots : X^n$ that one and only one of $X^1, \ldots, X^n$ be true.

If $X^1, \ldots, X^n$ are local sentences, then for the construction of locally true $X^1 \cdot \ldots \cdot X^n$ and $X^1 : \ldots : X^n$ observance of identity of time and place is required in addition to what was said above for universal $X^1, \ldots, X^n$.

If at least one of $X^1, \ldots, X^n$ is local then a universally true $X^1 \cdot \ldots \cdot X^n$ is not possible, as is evident from the definitions. If all $X^1, \ldots, X^n$ are local, then a universally true $X^1 : \ldots : X^n$ is possible as a tautology.

When, however, $X^1 : \ldots : X^n$ is not a tautology, then it cannot be universally true for the following reason: the objects spoken of in X^i are empirical; they do not all exist at a given time or in a given place; therefore, all X^i are unverifiable; this means that our sentence is unverifiable; what is more, any sentence of this type may turn out to be indeterminate for the same reason.

As we can see, the construction of true sentences in sentential logic is of no great interest. It becomes important only in reference to certain sentences which sentential logic considers elementary.

9. TERMS

Terms with the signs $\sim$, $\cdot$, $:$, ... are possible:

1) $\sim t \rightleftharpoons$ "An object which is not designated by the term t";

2) $t^1 \cdot t^2 \rightleftharpoons$ "An object which is designated by each of the terms t^1 and t^2"; similarly for $t^1 \cdot t^2 \cdot \ldots \cdot t^n$;

3) $t^1 : t^2 \rightleftharpoons$ "An object which is designated by one and only one of the terms t^1 and t^2"; similarly for $t^1 : t^2 : \ldots : t^n$.

4) $t^1 \vee t^2 \rightleftharpoons$ "An object which is designated by at least one of the terms t^1 and t^2; similarly for $t^1 \vee t^2 \vee \ldots \vee t^n$.

5) Similar definitions can be provided for terms with other signs derived from $\sim$, $\cdot$, $:$, $\vee$.

These terms are also used in a somewhat different sense, namely as "not...", "each of...", "one and only one of...". We will designate them with the symbols $(\bar{t})$, $(\cdot t^1, t^2)$, $(: t^1, t^2)$, etc. We formulate their properties below.

CHAPTER SIX

THE GENERAL THEORY OF LOGICAL ENTAILMENT

1. THE PROBLEM OF LOGICAL ENTAILMENT

Logical entailment is a key notion in logic. All other problems of logic center on the problem of logical entailment. Below we will examine the basic problems involved in the notion of logical entailment at the level of a general theory of logical entailment. This involves the rules of logical entailment for sentences with structures such as those examined in the previous chapter, and only for them.

There is a prejudice that there is some single, unchanging, "natural", "basic", etc., logical entailment and all logic has to do is to find the most accurate and complete (adequate) description for it. This view is based on the well-known fact that investigators can reason without having studied logic, that logical entailment led to great discoveries even in periods of weak logical development, and in our time similar progress is being made independently of progress in logic.

But it is precisely the efforts to find the most adequate description of logical entailment in contemporary logic that have destroyed this prejudice. As a matter of fact, there is no single, perfect, "natural", etc., logical entailment which simply has not been adequately described up to now. There are spontaneously arising manners of operating with sentences and terms and their primitive (intuitive) understanding. The task of logic is to perfect and make more precise the intuitive understanding of such ways of operating, to complete their differentiation, to explain their interconnections, etc. Logic has in fact now come to recognize different forms of logical entailment. No one of them can be considered more "basic" than another. In a certain sense they are all equal. The problem of the most adequate description of logical entailment is no longer a matter of finding some logical system as the final and unique theory of logical entailment but of constructing different types of logical systems, applicable to various intuitive premisses and the investigation of their properties and relations.

2. CLASSICAL THEORY OF ENTAILMENT

In contemporary logic the first form of the theory of logical entailment is the classical (Frege, Russell), the essence of which is the following: one assumes a functionally complete, two-valued sentential logic or a deductively equivalent classical sentential calculus with material implication. The tautologies of two-valued sentential logic and the corresponding provable formulae of the classical sentential calculus are considered rules of logical entailment (also, the sentential formulae are taken as sentences and the sign of material implication as a sign of logical entailment). The intuitive premiss of such an interpretation of classical logic is the following understanding of logical entailment: for the latter it is necessary and sufficient that for a true premiss there will not be a false conclusion (so that there will not be a false consequent for a true antecedent).

3. NON-CLASSICAL THEORY OF ENTAILMENT

According to the classical theory of logical entailment, from a false sentence follows anything and a true sentence follows from anything ("paradoxes" of material implication). Lewis drew attention to the fact that such an interpretation of material implication as logical entailment does not correspond to its intuitive understanding. Logical entailment is, to his mind, narrower than material implication. Lewis constructed logical systems (systems of strict implication), excluding "paradoxes" similar to those of material implication. Many works were written on strict implication. There is a question: what right did Lewis and his followers have to regard material implication as a logical form greater in extension than logical entailment? It is obvious they based their claim on the intuitive understanding of logical entailment which includes more than the postulate of true conclusions following from true premisses.

But in the interpretation of strict implication as logical entailment *tout court* other "paradoxical" results appeared: any proposition follows from an impossible one; a necessary proposition follows from any proposition ("paradoxes" of strict implication). Some authors considered these results to be incompatible with the intuitive understanding of logical entailment. Beginning with Ackermann they constructed logical systems which excluded the "paradoxes" of strict implication: these are systems of strong implication.

However, the systems of strong implication did not solve the problem either. In particular the exclusion of some "paradoxes" involved the exclusion of a number of formulae about which intuition has no doubts.

We cannot deal more thoroughly here with the logical investigations which began with the works of Lewis. For many reasons the classical theory of logical entailment is more suitable than the theories of strong and strict implication. But its application to the solution of a series of problems of the theory of scientific knowledge leads to serious difficulties if it is considered as the general theory of deduction. Therefore, the establishment of the non-classical theories was fully justified.

Below we will present the theory which seems to us most suitable for our conception of scientific knowledge. We will be guided by the following considerations: first, it is necessary to formulate an intuitive understanding of logical entailment and to construct a corresponding logical system according to the notion of a "formalizing" theory of logical entailment. Otherwise it would be impossible to solve such problems as: 1) what assures us that exclusion of "paradoxical" formulae does not involve exclusion of intuitively acceptable ones? and 2) where is the guarantee that the exclusion of one "paradoxical" formula will not lead to inclusion of intuitively unacceptable ones? Second, the desired theory of logical entailment has to be constructed not as a replacement for the classical theory (which cannot be cast out as superfluous and unacceptable) but as a more fundamental logical theory. Classical logic has to be preserved not as the unique general theory of logical entailment or as one of its variants but as its fragment.

4. THE GENERAL THEORY OF LOGICAL ENTAILMENT

The rules of logical entailment are formed as described in the previous chapter and constitute a sort of summary (or synthetic) definition of the sense of the logical signs, $\sim$, $\cdot$, $:$, and other derived signs. The particularity of the situation is that the defined objects (the logical signs) are in a sense established by the definitions and variations in the latter lead to variations in the objects themselves.

We will use the following designations:

1) $X \vdash Y$ for "Y strongly follows from X"

2) $X \succ\!\!- Y$ for "Y weakly follows from X"

3) $X \Vdash Y$ for "Y maximally follows from X"
4) $X \gg\!\!- Y$ for "Y follows conversely from X".

We adduce the following a priori definitions of these forms of entailment:

D1. $X \succ\!\!- Y$ is an entailment satisfying $X \not\triangleright Y$.

D2. $X \vdash Y$ is an entailment satisfying $X \not\!\not\triangleright Y$.

D3. $X \gg\!\!- Y$ is an entailment satisfying $X \not\triangleright Y$, with the following limitations: X contains only those elementary sentences which occur in Y.

D4. $X \Vdash Y$ is an entailment satisfying $X \not\!\not\triangleright Y$ with the limitation: X and Y contain identical elementary sentences.

Other forms of entailment are possible. But they are not interesting for us here. As a matter of fact, from the point of view of our basic theme both maximal entailment and converse entailment are not important. They are only constrictions of the corresponding strong and weak entailments.

We would like to call attention to the fact that $X \vdash Y$, $X \succ\!\!- Y$, etc., cannot be considered sentences formed from the sentences X and Y. They are only sentences formed from the term-subjects $[X]$ and $[Y]$, term-predicates $\vdash$, $\succ\!\!-$, etc. ("The second follows from the first"). They are elementary sentences.

D5. If $X \vdash Y$, then X is a premiss and Y is a conclusion. Similarly for the other forms of entailment.

The rules of logical entailment for sentences formed from elementary sentences and the logical signs $\sim$, $\cdot$, : (and the logical signs which they define) form the general theory of logical entailment. One does not take the structure of the elementary sentences into account. Therefore, the entailment sign occurs only once (it does not occur in both premiss and conclusion) in the assertions of the general theory of logical entailment of the type $X \vdash Y$, $X \succ\!\!- Y$, etc.

T1. If $X \succ\!\!- Y$, then in X and Y there is at least one identical elementary sentence (according to *D1* and *D2*V7). If $X \vdash Y$, then Y does not contain elementary sentences lacking in X (according to *D2* and *D1*V7).

T2. $(X \gg\!\!- Y) \rightarrow (X \succ\!\!- Y), (X \Vdash Y) \rightarrow (X \vdash Y)$

D6. If $X \vdash Y$ and $Y \vdash X$ then we will abbreviate as $X \dashv\vdash Y$.

T3. $(X \vdash Y) \rightarrow (X \succ\!\!- Y)$

T4. $\sim(X \succ\!\!- Y) \rightarrow \sim(X \vdash Y)$

T5. $(X \succ\!\!- Y)\cdot([X] \leftarrow v^1) \rightarrow ([Y] \leftarrow v^1)$

T6. $(X \succ\!\!- Y)\cdot([\sim Y] \leftarrow v^1) \rightarrow ([\sim X] \leftarrow v^1)$

T7. Similarly *T5* and *T6* for strong implication.

T8. A logical theory satisfying

$$(X \mathrel{\times\!\!\!\rhd} Y) \rightarrow (X \succ\!\!- Y) \quad \text{and} \quad (X \succ\!\!- Y)\cdot(Y \succ\!\!- Z) \rightarrow (X \succ\!\!- Z)$$

is impossible.

It is easy to see that

$$p \mathrel{\times\!\!\!\rhd} (p \vee q), (\sim p\cdot p) \mathrel{\times\!\!\!\rhd} (\sim p\cdot p) \vee (\sim p\cdot q)$$
$$(\sim p\cdot p) \vee (\sim p\cdot q) \mathrel{\times\!\!\!\rhd} \sim p\cdot(p \vee q), \sim p\cdot(p \vee q) \mathrel{\times\!\!\!\rhd} q\,.$$

And if we accept the stipulations of *T8*, we get

$$\sim p\cdot p \succ\!\!- q\,,$$

which does not satisfy *D1.*

It is clear that in the construction of the theory of weak entailment some of the four assertions above cannot be included among the rules of logical entailment (as is done, for example, by Ackermann), or the rule of transitivity has to be limited in some way.

A set of assertions of the type $X \vdash Y$, $X \succ\!\!- Y$, etc., forms the general theory of logical entailment. They are obtained by virtue of the principles:

1) If identical elementary sentences occur in X and Y and $X \equiv Y$, then $X \dashv\vdash Y$.

2) If identical elementary sentences occur in X and Y and $X \approx Y$, then $X \dashv\vdash Y$.

3) If $X \mathrel{\times\!\!\!\rhd} Y$, then $X \succ\!\!- Y$.

4) If $X \mathrel{\#\!\!\!\rhd} Y$, then $X \vdash Y$.

After a sufficiently complete list of such assertions has been compiled (built up in some way), the process can be "inverted". These assertions can be viewed as definitions of the logical signs according to the principle: these logical signs have the properties fixed in these assertions; these logical signs are invented so that $X \vdash Y$, $X \succ\!\!- Y$, etc. It is just such a listing of assertions which forms the so-called intuitive base of the axiomatic theory of logical entailment.

The following assertions (the basic principles of deduction) hold for all forms of logical entailment:

1) If Y follows from X and X is true, then Y is true (cf. *T5* and *T7* above).

2) If Y follows from X and Y is not true, then X is not true (cf. *T6* and *T7* above).

The basic principles of deduction are not included among the rules of deduction itself. Assertions like

$$(X \vdash Y) \cdot X \vdash Y \quad \text{and} \quad (X \vdash Y) \cdot \sim Y \vdash \sim X$$

are redundant since their use in some way requires the basic principle of deduction, i.e., one has to recognize the truth of

$$(X \vdash Y) \cdot X \quad \text{and} \quad (X \vdash Y) \cdot \sim Y;$$

and once these are accepted, one accepts the truth of X (correspondingly of $\sim Y$) and one accepts $X \vdash Y$; in this case the basic principles of deduction are enough for the recognition of the truth of Y (correspondingly of $\sim X$).

5. THE INTUITIVE THEORY OF LOGICAL ENTAILMENT

Thanks to the investigation we talked about above we can construct the following system Z^0 which defines the sense of the logical signs $\sim$, $\cdot$, and $:$, and effects the first description of the rules of strong logical entailment.

1. $X \vdash X$
2. $X \dashv \vdash \sim \sim X$
3. $X \cdot Y \vdash X$
4. $X \cdot Y \vdash Y$
5. $X \cdot Y \vdash Y \cdot X$
6. $X^1 \cdot \ldots \cdot X^n \vdash X^i \quad (i = 1, \ldots, n)$
7. $X^1 \cdot \ldots \cdot X^n \vdash X^i \cdot \ldots \cdot X^k \quad (k = 1, \ldots, n)$
8. $X^1 \cdot \ldots \cdot X^n \vdash X_1 \cdot \ldots \cdot X_n$,

where the conclusion differs from the premisses only by another arrangement of the sentences.

9. $\sim (X \cdot Y) \dashv \vdash (X \cdot \sim Y) : (\sim X \cdot Y) : (\sim X \cdot \sim Y)$
10. $\sim (X^1 \cdot \ldots \cdot X^n) \dashv \vdash Y^1 : \ldots : Y^k$,

where $Y^1, \ldots, Y^k$ are all possible sentences which differ from $X^1 \cdot \ldots \cdot X^n$

only by the presence of one and only one sign $\sim$ before at least one of $X^1,\ldots, X^n$.

11. $X:Y \vdash Y:X$
12. $X^1:\ldots:X^n \vdash X_1:\ldots:X_n$
13. $X:Y \dashv\vdash (X\cdot\sim Y):(\sim X\cdot Y)$
14. $X^1:\ldots:X^n \dashv\vdash Y^1:\ldots:Y^n$,

where $Y^1,\ldots, Y^n$ are all possible sentences which differ from $X^1\cdot\ldots\cdot X^n$ by the presence of one and only one $\sim$ before all $X^1,\ldots, X^n$ except one.

15. $(X:Y)\cdot X \vdash \sim Y$
16. $(X^1:X^2:\ldots:X^n)\cdot X^1 \vdash \sim X^2\cdot\ldots\cdot\sim X^n$
17. $(X^1:\ldots:X^n)\cdot(X^1:\ldots:X^i) \vdash \sim X^{i+1}\cdot\ldots\cdot\sim X^n$
18. $(X^1:\ldots:X^n)\cdot(X^1:\ldots:X^{n-1}) \vdash \sim X^n$
19. $(X:Y)\cdot\sim X \vdash Y$
20. $(X^1:X^2:\ldots:X^n)\cdot\sim X^1 \vdash X^2:\ldots:X^n$
21. $(X^1:\ldots:X^n)\cdot\sim X^1\cdot\ldots\cdot\sim X^i \vdash X^{i+1}:\ldots:X^n$
22. $(X^1:\ldots:X^n)\cdot\sim X^1\cdot\ldots\cdot\sim X^{n-1} \vdash X^n$
23. $\sim(X:Y) \dashv\vdash (X\cdot Y):(\sim X\cdot\sim Y)$
24. $\sim(X^1:\ldots:X^n) \dashv\vdash Y^1:\ldots:Y^k$,

where $Y^i,\ldots, Y^k$ is a set of sentences which includes $X^1\cdot\ldots\cdot X^n$ and all possible sentences which differ from it by the presence of one and only one $\sim$ before all $X^1,\ldots, X^n$ or before i $(1\leqslant i\leqslant n-2)$ of them (i.e., $\sim$ is lacking before at least two of $X^1,\ldots, X^n$ or is present before all).

25. $(X:Y)\cdot Z \dashv\vdash (X\cdot Z):(Y\cdot Z)$
26. $(X^1:\ldots:X^n)\cdot Y \dashv\vdash (X^1\cdot Y):\ldots:(X^n\cdot Y)$
27. $(X^1:X^2)\cdot(Y^1:Y^2) \dashv\vdash (X^1\cdot Y^1):(X^1\cdot Y^2):(X^2\cdot Y^1):(X^2\cdot Y^2)$
28. $(X^1:\ldots:X^n)\cdot(Y^1:\ldots:Y^m) \dashv\vdash (X^1\cdot Y^1):\ldots:(X^1\cdot Y^m):\ldots:$
    $:(X^n\cdot Y^1):\ldots:(X^n\cdot Y^m)$
29. $X\cdot Y\cdot Z \dashv\vdash X\cdot(Y\cdot Z)$
30. $X^1\cdot X^2\cdot\ldots\cdot X^n \dashv\vdash Z$,

where the conclusion differs from the premisses only by the arrangement of parentheses.

31. $X^1:X^2:X^3 \vdash X^1:(X^2:X^3)$
32. $X^1:\ldots:X^n \vdash Z$,

where the conclusion differs from the premisses only by the arrangement of parentheses.

33. $Y \vdash X^1 : X^2 : \ldots : X^n$,

where each of $X^1, \ldots, X^n$ is either $\alpha^1 p^1 \cdot \ldots \cdot \alpha^m p^m$ (where $\alpha^1, \ldots, \alpha^m$ indicates the presence or absence of $\sim$ and all $\alpha^{i1} p^{i1} \cdot \ldots \cdot \alpha^{im} p^{im}$ are pairwise different) or $\sim p \cdot p \cdot Z$, and Y differs from $X^1 : X^2 : \ldots : X^n$ only by the arrangement of parentheses.

The other logical signs ($\vee$, $|$, $\supset$ etc.) are introduced by assertions of the type $X \equiv Y$. One can formulate assertions of the type $X \dashv\vdash Y$ for them. For example,

$$X \vee Y \dashv\vdash \sim(\sim X \cdot \sim Y), X \supset Y \dashv\vdash \sim(X \cdot \sim Y).$$

If throughout the system Z^0 we put $\succ\!-$ for $\vdash$ and add

34. $\sim X \succ\!- \sim(X \cdot Y)$,

we get the system Z^{01} which provides the first ("intuitive") definition of logical entailment in the weakened sense.

This enumeration of assertions can be extended. But this is fully taken care of by deduction, for which it is enough to admit a series of meta-assertions: that it is permitted to put any sentential structures for p, q, r, $p^1, p^2, \ldots$, to apply the rule of transitivity, etc. And this is preliminary to axiomatization, in addition to the fact that some of the above assertions can be obtained from others with the help of such permissible transformations.

An axiomatization of the general theory of logical entailment ought to satisfy the conditions:

1) in axiomatic construction all the above assertions should be provable; the construction ought to correspond to the intuitive understanding of logical entailment;

2) in axiomatic construction not just any assertions should be provable (outside of those introduced above) but only those which satisfy the requirements I.

Axiomatization can be carried out in different ways. This is due to the fact that in axiomatization convenience is of primary concern; one and the same result can be obtained in different ways; the effort to be complete relative to some requirements can mean deviation from other requirements (the admission of "paradoxical" cases); fulfilling some requirements can mean violating some others, etc. Axiomatizations differ not only in form but also in extension of provable assertions.

6. DEGENERATE ENTAILMENT

There are sentences which are true exclusively in function of their logical structure and of statements accepted in logic. These are called logically true sentences. We will represent them with symbols of the type

$$\vdash X.$$

Logically true sentences can be regarded as conclusions of an empty set of premisses – i.e., as degenerate entailment. All tautologies are logically true sentences.

T1. If X is a tautology, then $\vdash X$.

7. QUASI-ENTAILMENT

D1. If $X \cdot Z \vdash Y$ and $\vdash Z$, then we will say that X quasi-entails Y. We will write quasi-entailment as

$$X \rightarrowtail Y.$$

8. REASONING AND ENTAILMENT

One has to distinguish the use of the rules of logical entailment in reasonings from the determination as to whether or not a logical entailment takes place. Sentence Y logically follows from sentence X if and only if there is a rule (an assertion of logic) according to which a sentence with a structure like that of Y logically follows from a sentence with a structure like that of X. If this is the case, then we obtain "Y follows from X" and nothing more.

In addition to the establishment of the fact of Y following from X, a reasoning includes the following operations:

1) recognition of the truth of X; "detachment" of X and transition to Y, i.e., recognition of the truth of Y and use of it in further operations, if there are any;

2) recognition of the fact that Y is not true; "detachment" of Y and recognition that $\sim X$ is true (X is not true), i.e., a transition to $\sim X$ and use of it in further operations, if there are any.

The final result of the process of reasoning is the acceptance of some sentence Y on the basis that some other sentences $X^1, \ldots, X^n$ are accepted, and the resultant sentence is obtained from them according to the rules of logical entailment. No matter how involved and complicated a reasoning process may be it can always be presented (in principle) in the following form:

1) Y logically follows from $X^1 \cdot \ldots \cdot X^n$.
2) $X^1, \ldots, X^n$ are all true.
3) This means that Y is true.

That the construction of the first assertion (i.e., the enumeration of all premisses for accepting Y) is not always possible and expedient is another matter.

9. SENTENCES ABOUT ENTAILMENT

A sentence which talks about the fact that $X \vdash Y$ is not a complex sentence formed from X and Y. It is elementary from the point of view of the general theory of logical entailment. Its subject is a pair of terms ($[X]$, $[Y]$) and its predicate is $\vdash$ ("From the first the second logically follows"). Relative to X and Y it is a meta-sentence. The properties of the predicate $\vdash$ are partially determined in the general theory of logical entailment and partially determined in other branches of logic (see below). This all applies to other forms of entailment, too. The definitions of predicates of entailment are constructed by enumerating the cases where some sentences follow from others. Predicates of entailment are special cases of logical predicates which are defined in logic. The truth-values of the sentence $X \vdash Y$ are defined as follows:

D1. $[X \vdash Y]$ is true, if and only if there is the corresponding rule of logical entailment.

D2. Similarly for $X \rightarrowtail Y$ and other forms.

Assertions on the dependence of logical entailment for some sentences on logical entailment for other sentences are obtained as derivative rules of inference. For example,

$$(X \vdash Y \cdot Z) \rightarrow (X \vdash Y) \cdot (X \vdash Z)$$
$$(X \vdash Y) \rightarrow (X \cdot Z \vdash Y),$$

etc., will be valid as derivative rules. This means that the rules of inference from sentences which contain the predicate of logical entailment are obtained as derivative.

CHAPTER SEVEN

FORMALIZATION OF THE GENERAL THEORY OF LOGICAL ENTAILMENT

1. STRONG LOGICAL ENTAILMENT

The system of strong logical entailment S^1 has the following form.

Alphabet:

1) the letters p, q and r, with or without superscripts, are sentential variables;

2) $\cdot$, $:$, $\sim$ are sentential constants (corresponding to "and", strict "or" and "not"; "and" is also read as "every one of..."; strict "or" is read as "one and only one of...");

3) $\vdash$ is the sign of strong entailment ("from ... strongly follows ...");

4) brackets and points are the bounds of formulae, providing the rules for reading them.

D1. The sentential formula:

1) sentential variables are sentential formulae;

2) if X is a sentential formula, $\sim X$ is a sentential formula;

3) if $X^1, X^2, \ldots, X^n$ $(n \geqslant 2)$ are sentential formulae, then $(X^1 \cdot X^2)$, $(X^1 : X^2)$, $(X^1 \cdot X^2 \cdot \ldots \cdot X^n)$ and $(X^1 : X^2 : \ldots : X^n)$ are sentential formulae;

4) something is a sentential formula only by virtue of 1 to 3.

D2. $X \vdash Y$ is a formula of strong logical entailment if and only if X and Y are sentential formulae.

D3. Sentential formulae and the formulae of strong entailment are formulae.

D4. A sentential formula occurs (is an occurrence) in a formula in the following cases and only in these:

1) the sentential formula X occurs in X, $\sim X$, $(X \cdot Y)$, $(Y \cdot X)$, $(X : Y)$, $(Y : X)$, $(X \cdot Y^1 \cdot \ldots \cdot Y^n)$, $(Y^1 \cdot \ldots \cdot Y^n \cdot X)$, $(Y^1 \cdot \ldots \cdot X \cdot \ldots \cdot Y^n)$, $(X : Y^1 : \ldots : Y^n)$, $(Y^1 : \ldots : Y^n : X)$, $(Y^1 : \ldots : X : \ldots : Y^n)$;

2) if the sentential formula X occurs in the sentential formula Y and Y occurs in the formula Z, then X occurs in Z.

According to *D4* the formula $(X^1 \cdot X^2)$ does not occur in $(X^1 \cdot X^2 \cdot X^3)$, the formula $(X^1:X^2)$ is not an occurrence in the formulae $(X^1:X^2:X^3)$, $(X^3:X^1:X^2)$, etc.

D5. $(X \supset Y)$ is shorthand for $\sim(X \cdot \sim Y)$. The $\supset$ is the sign of material implication. It is read exclusively as indicated in this definition.

D6. $(X \vee Y)$ is shorthand for $\sim(\sim X \cdot \sim Y)$; $X^1 \vee X^2 \vee \ldots \vee X^n$ is shorthand for $\sim(\sim X^1 \cdot \sim X^2 \cdot \ldots \cdot \sim X^n)$.

For the sake of simplicity we will omit brackets in many cases, assuming that $\cdot$ binds more strongly than $:$ and both of them more strongly than $\vdash$. The sign $\cdot$ will be omitted; the formulae it joins will be written in series without any interval. If $X \vdash Y$ and $Y \vdash X$, then for brevity we will write $X \dashv\vdash Y$.

The axioms of S^1:

A1. $p \dashv\vdash \sim\sim p$
A2. $pq \vdash p$
A3. $pq \vdash qp$
A4. $pqr \dashv\vdash p(qr)$
$p^1p^2 \ldots p^n \dashv\vdash X$,

where X differs from $p^1p^2 \ldots p^n$ by some arrangement of parentheses satisfying the definition of a sentential formula.

A5. $\sim(pq) \dashv\vdash \sim pq : p \sim q : \sim p \sim q$
A6. $\sim(p:q) \dashv\vdash pq : \sim p \sim q$
$\sim(p^1:p^2:p^3) \dashv\vdash p^1p^2p^3 : p^1p^2 \sim p^3 : p^1 \sim p^2p^3 :$
$: \sim p^1p^2p^3 : \sim p^1 \sim p^2 \sim p^3$
. . . .
$\sim(p^1:p^2:\ldots:p^n) \dashv\vdash X^1:\ldots:X^k$,

where $X^1,\ldots, X^k$ is a set of formulae including $(p^1p^2 \ldots p^n)$ and all possible formulae which differ from it by the presence of one and only one $\sim$ before all $p^1, p^2, \ldots, p^n$ or before i of them, where $1 \leqslant i \leqslant n-2$.

A7. $p^1:p^2:\ldots:p^n \vdash X$,

where X differs from $p^1:p^2:\ldots:p^n$ only by some arrangement of parentheses which satisfies the definition of a sentential formula.

A8. $Y \vdash X^1:X^2:\ldots:X^n$,

where each of $X^1, \ldots, X^n$ is either $\alpha^1 p^1 \cdot \ldots \cdot \alpha^m p^m$ (where $\alpha^1, \ldots, \alpha^m$ indicates the presence or absence of $\sim$ and all $\alpha^{i1} p^{i1} \cdot \ldots \cdot \alpha^{im} p^{im}$ are pairwise different) or $\sim ppZ$, and Y differs from $X^1 : X^2 : \ldots : X^n$ only by an arrangement of parentheses.

A9. $pr:qr \vdash (p:q)\, r$
$p^1 q : p^2 q : p^3 q \vdash (p^1 : p^2 : p^3)\, q$
. . . .
$p^1 q : p^2 q : \ldots : p^n q \vdash (p^1 : p^2 : \ldots : p^n)\, q$
A10. $(p:q)\, r \vdash pr:q$
$(p^1 : p^2 : p^3)\, q \vdash p^1 q : p^2 : p^3$
. . . .
$(p^1 : p^2 : \ldots : p^n)\, q \vdash p^1 q : p^2 : \ldots : p^n$
$(p^1 : p^2)(q^1 : q^2) \vdash p^1 q^1 : p^1 q^2 : p^2$
. . . .
$(p^1 : \ldots p^n)(q^1 : \ldots : q^m) \vdash p^1 q^1 : \ldots : p^1 q^m : p^2 : \ldots : p^n .$

Axioms *A4*, *A6*, *A9* and *A10* can be regarded as sets of axioms. But the last lines in them can be regarded as descriptions of general cases and the preceding lines as special cases, explaining the general cases.

D7. Provable formulae:

1) the axioms of S^1 are provable formulae in S^1 (or, provable in S^1);

2) if $X^1 \vdash X^2$ is provable in S^1 and $Y^1 \vdash Y^2$ is obtained from $X^1 \vdash X^2$ by putting a sentential formula in place of a sentential variable wherever this variable occurs in $X^1 \vdash X^2$, then $Y^1 \vdash Y^2$ is provable in S^1;

3) if $X \vdash Y$ and $Y \vdash Z$ are provable in S^1, then $X \vdash Z$ is provable in S^1;

4) if $X \vdash Y$ and $X \vdash Z$ are provable in S^1, then $X \vdash YZ$ is provable in S^1;

5) if $X^1 \vdash X^2$ and $X^2 \vdash X^1$ are provable in S^1 and Y^2 is obtained from Y^1 by substituting for the occurrence of X^1 in Y^1 the sentential formula X^2, then $Y^1 \vdash Y^2$ is provable in S^1;

6) a formula is provable in S^1 only by virtue of 1)–5).

Points 2 to 5 of *D7* are the rules of inference of S^1 (the substitution of sentential variables is *R1*, transitivity is *R2*, combination of consequents is *R3*, substitution of equivalents is *R4*). They can also be written as follows:

R1. If $X^1 \vdash X^2$, then $Y^1 \vdash Y^2$, where $Y^1 \vdash Y^2$ is obtained from $X^1 \vdash X^2$ by putting a sentential formula for a sentential variable wherever the latter occurs in $X^1 \vdash X^2$.

R2. If $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.

R3. If $X \vdash Y$ and $X \vdash Z$, then $X \vdash YZ$.

R4. If $X^1 \dashv\vdash X^2$, then $Y^1 \vdash Y^2$, where Y^2 is obtained from Y^1 by putting the sentential formula X^2 for the occurrence of X^1 in Y^1.

The definition *D7* can here be modified as follows: the formula $X \vdash Y$ is provable in S^1 if and only if it is an axiom of S^1 or is obtained from provable formulae of S^1 according to the rules *R1–R4.*

It follows from *D7* that no sentential formula is provable in S^1.

We adduce the following semantic interpretation of the signs of S^1:

1) sentential variables take the values 1 and 0;

2) $X^1 \cdot \ldots \cdot X^n$ $(n \geqslant 2)$ takes the value 1 if all $X^1, \ldots, X^n$ take the value 1, and the value 0 in all other cases;

3) $X^1 : \ldots : X^n$ takes the value 1, if one and only one of the formulae of $X^1, \ldots, X^n$ take the value 1, and the value 0 in all other cases;

4) $\sim X$ takes the value 1, if X takes the value 0, and the value 0 if X takes the value 1;

5) $X \vdash Y$ is equivalent to $X \supset Y$ or, what is the same, equivalent to $\sim (X \sim Y)$;

6) a formula is called a tautology if and only if it takes the value 1 for any combination of values of its variables.

The following theorems hold for S^1.

T1. If $X \vdash Y$ is provable in S^1, then $X \vdash Y$ is a tautology (theorem of consistency).

T2. If $X \vdash Y$ is provable in S^1, then there occur in Y only those sentential variables which occur in X (theorem of non-paradoxicality).

T3. If $X \vdash Y$ is a tautology and in Y there are no sentential variables lacking in X, then $X \vdash Y$ is provable in S^1 (completeness theorem).

The proof of theorems *T1–T3* has been given by G. A. Smirnov (see Appendix). He has also proved the independence of the axioms and rules of inference of S^1.

2. ANOTHER VARIANT OF THE SYSTEM OF STRONG ENTAILMENT

The system S_1 of strong entailment is distinguished from S^1 only by the following.

Instead of the constant : we use $\vee$ throughout (connective disjunction).

This is interpreted as: $X^1 \vee \ldots \vee X^n$ $(n \geqslant 2)$ takes the value 0 if all $X^1, \ldots, X^n$ take the value 0, and the value 1 for all other cases. Instead of *D6* we accept the definition:

D6. $X{:}Y$ is an abbreviation of $X \sim Y \vee \sim XY$; $X^1{:}X^2{:}\ldots{:}X^n$ is an abbreviation for $X^1 \sim X^2 \ldots \sim X^n \vee X^2 \sim X^1 \ldots \sim X^n \vee \ldots \vee X^n \sim X^1 \ldots \sim X^{n-1}$. One adduces another system of axioms.

The axioms of S_1:

A1. $\sim \sim p \vdash p$
A2. $p \vdash \sim \sim p$
A3. $pq \vdash p$
A4. $pq \vdash qp$
A5. $p^1 p^2 \ldots p^n \vdash p^1 (p^2 \ldots p^n)$ $(n \geqslant 3)$
$p^1 \ldots p^{n-1} p^n \vdash (p^1 \ldots p^{n-1}) p^n$ $(n \geqslant 3)$
$p^1 \ldots p^i q^1 \ldots q^k r^1 \ldots r^l \vdash p^1 \ldots p^i (q^1 \ldots q^k) r^1 \ldots r^l$ $(i \geqslant 1, k \geqslant 2, l \geqslant 1)$
A6. $p^1 (p^2 \ldots p^n) \vdash p^1 p^2 \ldots p^n$
$(p^1 \ldots p^{n-1})\, p^n \vdash p^1 \ldots p^{n-1} p^n$
$p^1 \ldots (q^1 \ldots q^k) \ldots r^l \vdash p^1 \ldots q^1 \ldots q^k \ldots r^l$
A7. $(p \vee q)\, r \vdash pr \vee q$
A8. $pr \vee qr \vdash (p \vee q)\, r$
A9. $\sim (pq) \vdash \sim p \vee \sim q$
$\sim (p^1 p^2 \ldots p^n) \vdash \sim p^1 \vee \sim p^2 \vee \ldots \vee \sim p^n$
A10. $\sim p \vee \sim q \vdash \sim (pq)$
$\sim p^1 \vee \sim p^2 \vee \ldots \vee \sim p^n \vdash \sim (p^1 p^2 \ldots p^n)$
A11. $pq \vee r \vdash (pq \vee r)(q \vee \sim q)$.

Theorems analogous to *T1–T3* of S^1 are true for S_1.

For the proof of *T3* we introduce the following definitions.

*D*1.* We will say that the sentential formula X occurs in a normal form if and only if it has the form $Y^1 \vee \ldots \vee Y^n$ $(n \geqslant 2)$ and satisfies the following conditions: 1) every formula of Y^i is a formula $\alpha^1 p^1 \cdot \ldots \cdot \alpha^m p^m$ $(m \geqslant 1)$, where $p^1, \ldots, p^m$ are all variables occurring in X; $\alpha^1, \ldots, \alpha^m$ indicates the presence or absence of negation and all $\alpha^1 p^1, \ldots, \alpha^m p^m$ are pairwise different; 2) if a variable p^i occurs in some Y^i without negation, then among $Y^1, \ldots, Y^n$ is found a Y^k (not necessarily other) in which $\sim p^i$ occurs; 3) all $Y^1, \ldots, Y^n$ are pairwise different.

*D*2.* We will say that $X \vdash Y$ occurs in normal form if and only if both

X and Y occur in normal form and the sets of variables occurring in them are identical.

The following lemmas are also necessary for the proof of *T3*.

L1. $X \dashv\vdash X(p \vee \sim p)$, where p occurs in X, is provable in S^1.

L2. $X \vee \sim ppY \dashv\vdash X$, where p occurs in X and in Y there are no variables lacking in X, is provable in S^1.

L3. For every formula $X \vdash Y$ one can find a formula $X^* \vdash Y^*$ in a normal form such that $X \dashv\vdash X^*$ and $Y \dashv\vdash Y^*$ are provable in S_1.

L4. If $X \vdash Y$ is a tautology, then $X^* \vdash Y^*$ is a tautology, where $X^* \vdash Y^*$ is the same as in *L3*.

L5. If $X^* \vdash Y^*$ is provable, then $X \vdash Y$ is provable, where $X^* \vdash Y^*$ is the same as in *L3*.

L6. If $X \vdash Y$ is a tautology and occurs in normal form, then $X \vdash Y$ is provable in S_1.

The proof of *L6*. Let X be $Z^1 \vee \dots \vee Z^k$ and Y be $Z_1 \vee \dots \vee Z_l$. Two cases are possible: 1) Y is a contradiction; 2) Y is satisfiable. Let us take the first case. If Y is a contradiction, then X is also a contradiction. This means that all Z^i and Z_j are contradictions. Let $V^1, \dots, V^m$ be all possible contradictions formed from variables occurring in X and Y such that $V^1 \vee \dots \vee V^m$ occurs in normal form. It is clear that all Z^i and Z_j are found among $V^1, \dots, V^m$. $V^1 \vee \dots \vee V^m \vdash V^1 \vee \dots \vee V^m$ is provable in S_1. According to *L2* we obtain that $V^1 \vee \dots \vee V^m \vdash Z_1 \vee \dots \vee Z_l$ and $Z^1 \vee \dots \vee Z^k \vdash V^1 \vee \dots \vee V^m$ are provable. This means that $Z^1 \vee \dots \vee Z^k \vdash Z_1 \vee \dots \vee Z_l$, i.e., $X \vdash Y$, is provable.

Two subcases are possible in the second case. In the first Y is not a tautology. If Y is satisfiable then $Z_{i1}, \dots, Z_{ir}$ $(r \geqslant 1)$ is satisfiable, where $Z_{i1}, \dots, Z_{ir}$ are any of $Z_1, \dots, Z_l$. Let it be the case that no formula of $Z_{i1}, \dots, Z_{ir}$ occurs in X. Then X has to be a contradiction (otherwise it could be true with a false Y). With $X \vdash X$, according to *L2* we obtain $X \vdash X \vee Y$ and $X \vdash Y$. Let $Z_{j1}, \dots, Z_{js}$ $(s \geqslant 1)$ be all of $Z_{i1}, \dots, Z_{ir}$ occurring in X. Since Y is not a tautology, other satisfiable Z^i should not occur in X. Hence all remaining Z^i are contradictions. We obtain $Z_{j1} \vee \dots \vee Z_{js} \vdash Z_{j1} \vee \dots \vee Z_{js}$, and according to *L2* we obtain $Z^1 \vee \dots \vee Z^k \vdash Z_{j1} \vee \dots \vee Z_{js}$, $Z_{j1} \vee \dots \vee Z_{js} \vdash Z_{j1} \vee \dots \vee Z_{js} \vee Z_1 \vee \dots \vee Z_l$, $Z_{j1} \vee \dots \vee Z_{js} \vdash Z_1 \vee \dots \vee Z_l$, $Z^1 \vee \dots \vee Z^k \vdash Z_1 \dots \vee Z_l$, i.e. $X \vdash Y$.

The second subcase of the second case: Y is a tautology. This means that all possible satisfiable formulae with the corresponding variables occur in Y. If not one of satisfiable Z_i occurs in X, then all $Z^1, \dots, Z^k$ are contradictions (this case has already been examined). If only one satisfiable Z_i occurs in X, this case, too, was examined above. Therefore, if $X \vdash Y$ is a tautology then $X \vdash Y$ is provable in S_1. Because of *L4* and *L5* our theorem *T3* will be true.

For systems S^1 and S_1 it follows from theorems *T1–T3* that they are equivalent in the following sense:

T4. If $X^1 \vdash X^2$ is provable in S^1 (in S_1) and $Y^1 \vdash Y^2$ is formed from $X^1 \vdash X^2$ by replacing all occurrences of X^1 and X^2 with the sign : (with the sign $\vee$) with formulae with the sign $\vee$ (with the sign :) in accordance with *D6* in S_1 (in S^1), then $Y^1 \vdash Y^l$ is provable in S_1 (in S^1).

Instead of *D6* one can add the axioms

$$p^1 \vee \dots \vee p^n \dashv\vdash \sim(\sim p^1 \cdot \dots \cdot \sim p^n),$$

to the axioms of S^1, and the axioms

$$\begin{aligned} &p^1 : p^2 \dashv\vdash p^1 \sim p^2 \vee \sim p^1 p^2 \\ &p^1 : p^2 : \dots : p^n \dashv\vdash p^1 \sim p^2 \dots \sim p^n \vee p^2 \sim p^1 \dots \sim p^n \vee \\ &\qquad\qquad\qquad\qquad \vee \dots \vee p^n \sim p^1 \dots \sim p^{n-1}. \end{aligned}$$

to the axioms of S_1.

In such a case *T4* can be written simply in the following form: if $X \vdash Y$ is provable in one of S^1 and S_1, it is provable in the other.

In what follows, all references to systems S^1 and S_1 will assume that the constants : and $\vee$ can both be used because of *D6* or because of the additional axioms (of course, the alphabet has to be widened accordingly and some additions made to *D1* and *D4*).

3. WEAKENED LOGICAL ENTAILMENT

The system S^2 of weakened logical entailment is formed from S^1 as follows.

The expression "strong logical entailment" is everywhere replaced by the expression "weakened logical entailment". We adduce the additional axiom

A11. $\sim p \vdash \sim (pq)$.

The rule *R2* takes the form:

R2. If $X \vdash Y$ and $Y \vdash Z$ and there occurs in X, Y and Z at least one identical sentential variable, then $X \vdash Z$.

We will not put another sign for $\vdash$ and will indicate in what follows the system in which it is used (if this is necessary).

T1. If $X \vdash Y$ is provable in S^2 then $X \vdash Y$ is a tautology (theorem of consistency).

T2. If $X \vdash Y$ is provable in S^2, then there occurs in X and Y at least one identical sentential variable (theorem of non-paradoxicality).

T3. If $X \vdash Y$ is a tautology and there occurs in X and Y at least one identical sentential variable, then $X \vdash Y$ is provable in S^2 (completeness theorem).

The proof of *T1–T3* is provided by G. A. Smirnov. He is also responsible for the idea of limiting the rule of transitivity in systems of weakened entailment. The system S_2 of weakened entailment is equivalent to S^2 obtained from S_1 by analogous modifications and additions. Theorems analogous to *T1–T3* are true in S_2. Their proof is provided by A. M. Fedina. The problem of independence for S^2 and S_2 has been investigated by G. A. Smirnov and E. A. Sidorenko. (For all of this, see the Appendix.)

4. MAXIMAL LOGICAL ENTAILMENT

The system S^3 of maximal logical entailment is formed from S^1 through substitution for *A2*:

A2. If only those variables which occur in X occur in Y, then $XY \vdash X$.

The system S_3 is formed from S_1 through a similar substitution of *A3*.

T1. The formula $X \vdash Y$ is provable in S^3 (and in S_3) if and only if $X \vdash Y$ is a tautology and the sets of sentential variables in X and Y are identical.

The proof of *T1* for S^3 is provided by G. A. Smirnov, and for S_3 by A. M. Fedina (see the Appendix).

5. CONVERSE LOGICAL ENTAILMENT

The system S^4 of converse logical entailment is obtained from S^3 by addition of the axiom

A11. $\sim p \vdash \sim (pq)$.

The system S_4 is formed from S_3 by addition of the axiom *A11.*

T1. The formula $X \vdash Y$ is provable in S^4 (in S_4) if and only if $X \vdash Y$ is a tautology and there occur in X only those variables which occur in Y.

The proof of *T1* for S^4 is provided by G. A. Smirnov and for S_4 by A. M. Fedina (see the Appendix).

6. DEGENERATE LOGICAL ENTAILMENT

The system S^5, including degenerate logical entailment, is obtained as follows. We adduce S^1 (or S_1) and the following additions to it.

D8. $\vdash X$ is a formula of degenerate entailment if and only if X is a sentential formula.

D9. A formula of degenerate entailment is a formula.

Supplementary axiom:

A11. $\vdash \sim(\sim pp)$.

Supplementary rule:

R5. If $X \vdash Y$ and $\vdash X$, then $\vdash Y$.

D10. $\vdash X$ is provable in S^5 if and only if it is an axiom or is obtained from provable formulae according to rule *R5*.

T1. $\vdash X$ is provable in S^5 if and only if X is a tautology.

From *T1* follows: if $\vdash X$ is provable in S^5, then X is provable in classical sentential calculus (with the corresponding logical constants) and vice versa.

R5 is enough for the completeness of S^5 in the sense of *T1*. In what follows we will expand the theory of logical entailment in such a way that it is necessary to expand the theory of degenerate entailment through the acceptance of the following rules:

R6. If $X \vdash Y$ and $\vdash \sim Y$, then $\vdash \sim X$.

R7. If $\vdash X$ and $\vdash Y$, then $\vdash XY$.

In *D10* we put "according to *R5–R7*" for "according to *R5*".

7. QUASI-ENTAILMENT

The system S^6 of quasi-entailment is obtained from S^5 through the following additions.

D11. $X \rightarrowtail Y$ is a formula of quasi-entailment if and only if X and Y are sentential formulae.

D12. A formula of quasi-entailment is a formula.

D13. The formula $X \rightarrowtail Y$ is provable in S^6 if and only if $XZ \vdash Y$ and $\vdash Z$ are provable in S^5.

Instead of *D13*, one can adduce the rule of inference:

R8. If $XZ \vdash Y$ and $\vdash Z$, then $X \rightarrowtail Y$.

Then *D13* takes the form: the formula $X \rightarrowtail Y$ is provable if and only if it is obtained from the formulae provable in S^5 according to rule *R8*.

T1. $X \rightarrowtail Y$ is provable in S^6 if and only if $X \supset Y$ is a tautology.

8. LOGICAL ENTAILMENT AND CLASSICAL SENTENTIAL CALCULUS

Because of *D2*VII1 expressions of the type

$$X \vdash (Y \vdash X), \quad X \vdash (\sim X \vdash Y), \quad (X \vdash Y)(Y \vdash Z) \vdash (X \vdash Z),$$

etc., containing two or more implication signs, are not formulae of S^i and S_i. Therefore, not every well-formed formula of the classical sentential calculus, of the type $X \supset Y$, has a corresponding formula of the type $X \vdash Y$ in our systems.

Because of the theorems of non-paradoxicality for systems S^i and S_i, not every formula of the type $X \supset Y$, provable in classical sentential calculus, has a corresponding formula $X \vdash Y$ in our systems. Thus, the formulae

$$X \vdash \sim (\sim YY), \quad X \vdash Y \vee \sim Y, \quad \sim XX \vdash Y$$

are unprovable for all S^i and S_i; the formula

$$X \vdash X \vee Y$$

is unprovable in systems of strong and maximal entailment; the formula

$$XY \vdash X$$

is unprovable in the system of converse entailment.

The system S^5 is equivalent to classical sentential calculus in the following sense: if $\vdash X$ is provable in S^5, then X is provable in classical sentential calculus (with the appropriate logical constants); and vice versa. This is a consequence of *T1*VII7.

9. PARADOXES OF ENTAILMENT

As was already indicated, the expressions

$$X \vdash (Y \vdash X) \quad \text{and} \quad X \vdash ({\sim} X \vdash Y)$$

are not formulae of our systems and formulae of the type

$$X \vdash {\sim}({\sim} YY), \quad X \vdash Y \vee {\sim} Y, \quad {\sim} XX \vdash Y$$

are unprovable. Therefore, paradoxes like those of material and strict implication do not happen here. And, the exclusion of paradoxes is accomplished without excluding intuitively non-paradoxical formulae: our systems are, in a sense, complete.

But, the following has to be kept in mind. The formulae α:

$$p \vdash {\sim} p{:}p, \quad p \vdash {\sim} p \vee p, \quad {\sim} pp \vdash p$$

are provable in S^1 (and S_1).

They are sometimes regarded as special cases of the "paradoxical" formulae β:

$$q \vdash {\sim} p{:}p, \quad q \vdash {\sim} p \vee p, \quad {\sim} pp \vdash q.$$

In this regard it should be noted that the expression "special case of a formula" is of itself devoid of meaning. Only the notion "special case of a provable formula" has sense when defined as follows: a formula X is a special case of a formula Y, provable in some logical system, if and only if X is obtained from Y through substitution for variables occurring in Y. From this point of view α are not a special case of β since the latter are not provable in S^1 (and S_1). And if such formulae are disliked for some reason, then they have to be formulated independently of β.

Of course, one could reject (for some reason) formulae which are provable in some system and construct a narrower calculus. However in this case one would simply have to enumerate the excluded formulae or point out some of their general structural characteristics. For example, it could be required that formulae of the type

$$p \vdash p{:}X, \quad p \vdash p{:}X^1{:}\ldots{:}X^n$$

be unprovable. However, in all such cases one cannot formulate *a priori* requirements which do not depend on the concrete structure of the formulae.

Formulae like α are the price one has to pay for the deductive method and for the completeness of scope of formulae of a certain type, in a given calculus.

Thus, the problem of logical entailment can be considered solved in principle since there are no other *a priori* criteria of meaningful sentential relations, which would be as developed and defined as relations of sets of variables occurring in premisses and conclusions. Relations between formulae according to length offer nothing here and to inquire into the concrete structure of formulae would mean to relinquish some *a priori* (intuitive) notion of logical entailment, which does not depend on the type of formulae and calculi; i.e., to avoid the problem.

Of course, it is possible to find other variants of logical systems, which are different from ours but equivalent to them. This, however, does not change the fact that the solution of the problem is basically achieved. Also possible are systems which differ from ours in respect to the extension of provable formulae. But this will be the solution of problems other than the problem of logical entailment.

We should also note that the systems of Ackermann, Anderson, Belnap and others – as concerns the defining of logical entailment – do not meet the above requirements; they are, in a sense, "improper" ("deformed", etc.) systems. This is no accident. Their authors tried to eliminate formulae of a certain type (the paradoxes of material and strict implication) and not to construct complete (in the above sense) systems, all the provable formulae of which meet strictly *a priori* requirements of meaning.

10. CLASSICAL AND NON-CLASSICAL SENTENTIAL RELATIONS

T1. If the formula $\vdash(X{:}\,Y)$ is provable in S^6, then the formulae $\vdash\sim(X\cdot Y)$, $\vdash\sim(\sim X\cdot\sim Y)$, $X\rightarrowtail\sim Y$, $Y\rightarrowtail\sim X$, $\sim X\rightarrowtail Y$, $\sim Y\rightarrowtail X$ are provable.

T2. The formula $X{:}\,Y{:}\,XY\dashv\vdash X{:}\,Y$ is provable in S^1. If one assumes $\vdash X{:}\,Y{:}\,XY$ to be provable, then $\vdash(X{:}\,Y)$ is provable in S^5.

T3. If $\vdash(X{:}\,Y{:}\sim X\cdot\sim Y)$, then $\vdash\sim(X\cdot Y)$, $X\rightarrowtail\sim Y$, $Y\rightarrowtail\sim X$, are provable according to S^6 but $\vdash\sim(\sim X\cdot\sim Y)$, $\sim X\rightarrowtail Y$, $\sim Y\rightarrowtail X$, are not provable.

The above cases are interesting for the analysis of possible relations between sentences. There are cases where the relation between two sentences A and B is such that $A{:}B$ is accepted (provable). Using *T1* we obtain the corresponding consequences from $A{:}B$, namely that

$$\sim(A\cdot B),\quad \sim(\sim A\cdot\sim B),\quad A\rightarrowtail\sim B,\quad B\rightarrowtail\sim A,$$
$$\sim A\rightarrowtail B,\quad \sim B\rightarrowtail A$$

are provable (accepted).

Where $A{:}B{:}AB$ is accepted there is nothing of special interest since it is equivalent to the acceptance of $A{:}B$ according to *T2*.

Acceptance of $A{:}B{:}\sim A\sim B$ is actually quite frequent in practical activity. It is interesting because in such a case $\sim(AB)$, $A\rightarrowtail\sim B$ and $B\rightarrowtail\sim A$ hold but $\sim(\sim A\cdot\sim B)$, $\sim A\rightarrowtail B$ and $\sim B\rightarrowtail A$ do not.

For example, let A be "N affirms that X", where X is some sentence; let B be "N affirms that $\sim X$". It is possible that N does not affirm X and does not affirm $\sim X$. If N is logical, then $\vdash\sim(A\cdot B)$, $A\rightarrowtail\sim B$ and $B\rightarrowtail\sim A$. But, from $\sim A$ it does not follow that B; from $\sim B$ it does not follow that A. Similarly if A is "N wishes that X" and B is "N wishes that $\sim X$".

The case considered in *T3* is interesting because we obtain three logically admissible possibilities using classical logic. It is not possible to obtain four or more logical possibilities for two sentences.

Let the fourth possibility be $\sim X\cdot\sim(\sim X\cdot\sim Y)$. But according to S^1, $\sim X\cdot\sim(\sim X\cdot\sim Y)\vdash Y$, i.e., this fourth possibility is reduced to one of the other three so that $X{:}Y{:}\sim X\sim Y{:}\sim X\sim(\sim X\sim Y)\vdash X{:}\sim X\sim Y$, i.e., the number of possibilities is in fact reduced. Similarly for $\sim Y\sim(\sim X\sim Y)$. Assumption of $\sim X\sim Y\sim(\sim X\sim Y)$ means assumption of contradiction, and according to S^1 $X{:}Y\sim X\sim Y{:}\sim X\sim Y\sim(\sim X\sim Y)\vdash X{:}Y{:}\sim X\sim Y$ is provable.

We will now examine analogous relations for any number of sentences.

T4. If $\vdash(X^1{:}\ldots{:}X^n)$ where $n\geqslant 2$, is provable, then according to S^6 $\vdash\sim(X^iX^k)$ (where $i\neq k$; $i=1,\ldots,n$; $k=1,\ldots,n$), $\vdash\sim(X^{i1}\cdot\ldots\cdot X^{ik})$ (where $X^{i1},\ldots,X^{ik}$ is any combination of different $X^1,\ldots,X^n$), $\vdash\sim(\sim X^1\cdot\ldots\cdot\sim\sim X^n)$, $X^i\rightarrowtail\sim X^k$, $\sim X^i\rightarrowtail X_1{:}\ldots{:}X_{n-1}$ (where $X_1,\ldots,X_{n-1}$ are all other of $X^1,\ldots,X^n$, different from X^i), etc., are provable.

T5. $X^1{:}\ldots{:}X^n{:}X^iX^k\dashv\vdash X^1{:}\ldots{:}X^n$ (where X^i and X^k are any pair of $X^1,\ldots,X^n$) is provable in S^1. If one accepts that $\vdash(X^1{:}\ldots{:}X^n{:}X^iX^k)$ is provable, then $\vdash(X^1{:}\ldots{:}X^n)$ is provable according to S^5. Similarly for any $X^{i1}\cdot\ldots\cdot X^{ik}$.

T6. The formula $X^1{:}\ldots{:}X^n{:}\sim X^i\sim X^k\vdash X^1{:}\ldots{:}X^n{:}\sim X^1\ldots\sim X^n$ is provable in S^1. Accepting $\vdash(X^1{:}\ldots{:}X^n{:}\sim X^i\sim X^k)$ we have to accept $\vdash(X^1{:}\ldots{:}{:}X^n{:}\sim X^1\cdot\ldots\cdot\sim X^n)$ according to S^5.

T7. The formula $X^1{:}\ldots{:}X^n{:}\sim X^{i1}\cdot\ldots\cdot\sim X^{ik}{:}\sim X_{i1}\cdot\ldots\cdot\sim X_{il}\vdash X^1{:}\ldots{:}{:}X^n$ is provable in S^1. Accepting $X^1{:}\ldots{:}X^n{:}\sim X^{i1}\cdot\ldots\cdot\sim X^{ik}{:}\sim X_{i1}\cdot\ldots\cdot\sim\sim X_{il}$, we have to accept $X^1{:}\ldots{:}X^n$. Similarly for $\sim X_1^{i1}\cdot\ldots\cdot\sim X_1^{ik}{:}\ldots{:}\sim\sim X_m^{i1}\cdot\ldots\cdot\sim X_m^{il}$, where $m\geqslant 2$.

T8. If $\vdash X^1 : \ldots : X^n : \sim X^1 \cdot \ldots \cdot \sim X^n$ is provable then according to S^6 $\vdash \sim (X^i X^k)$, $\vdash \sim (X^{i1} \cdot \ldots \cdot X^{ik})$, $X^i \rightarrowtail X^k$ are provable but $\vdash \sim (\sim X^1 \cdot \ldots \cdot \sim \sim X^n)$, $\sim X^i \rightarrowtail X_1 : \ldots : X_{n-1}$ are not provable.

Thus, if we have the sentences $A^1, \ldots, A^n$ $(n \geqslant 2)$, we can obtain only $n+1$ logically conceivable possibilities.

We introduce the definitions:

D1. The sentences $A^1, \ldots, A^n$ $(n \geqslant 2)$ occur in the classical relation if and only if $A^1 : \ldots : A^n$ is true.

D2. The sentences $A^1, \ldots, A^n$ occur in a non-classical relation if and only if $A^1 : \ldots : A^n : \sim A^1 \cdot \ldots \cdot \sim A^n$ is true and

$$\sim (A^1 : \ldots : A^n : \sim A^1 \cdot \ldots \cdot \sim A^n \vdash A^1 : \ldots : A^n).$$

Thus the syntactically non-classical case is distinguished from the classical only by the assumption of one additional possibility which is a conjunction of the negations of possibilities assumed by the classical. All other combinations are excluded.

11. NON-CLASSICAL CASES IN THE GENERAL THEORY OF DEDUCTION

The systems S_n^i are obtained through the following additions to S^i (to S_i).

Addition to the alphabet: $\urcorner$ for partial (or intrinsic) negation.

Addition to the definition of a sentential formula: if X is a sentential formula then $\urcorner X$ is a sentential formula.

Addition to the interpretation: if one of X and $\urcorner X$ has the value 1, the other has the value 0; but, if one of them has the value 0, the value of the other does not depend on that of the first.

Additional axioms:

1. $p \vdash \sim \urcorner p$
2. $\urcorner p \vdash \sim p$

T1. All formulae provable in S_n^i are tautologies.

T2. The formulae $\sim \urcorner p \vdash p$, $\vdash p \vee \urcorner p$ and $\vdash p : \urcorner p$ are not provable in S_n^i (since they are not tautologies).

The systems S^i are classical compared with S_n^i. The classical case is obtained by rejection of the additions made in this section or by adding the axiom

$$\sim p \vdash \urcorner p.$$

Here the axiom 1 is superfluous (dependent).

T3. The formula $p:\neg p:\sim p\sim\neg p\vdash p:\neg p$ is not provable in S_n^i (since it is not a tautology).

T4. The formula $\vdash p:\neg p:\sim p\sim\neg p$ is provable in S_n^5.

T5. The formulae $p:\neg p:p\neg p\vdash p:\neg p$, $p:\neg p\vdash p:\neg p:p\neg p$ are provable in S_n^1; the formulae $p:\neg p:p\sim\neg p\vdash\neg p$, $\neg p\vdash p:\neg p:p:\sim\neg p$, $p:\neg p::\sim p\neg p\vdash p$, $p\vdash p:\neg p:\sim p\neg p$ are provable in S_n^2.

Thus the non-classical case we have examined is the only possible one for two negations: only $\sim p\sim\neg p$ can expand the number of possibilities (according to *T3*); all other combinations do not increase the number of possibilities (according to *T5*).

For $\neg p$ there is the possible interpretation: $\neg p$ is equivalent to $\sim p\cdot\alpha$, where α is any sentential variable which does not occur in a formula in which $\neg p$ occurs.

The symbol $?p$ can be introduced as shorthand for $\sim p\sim\neg p$.

12. EXPANSION OF THE GENERAL THEORY OF LOGICAL ENTAILMENT

The systems examined above form the general theory of logical entailment (or the general theory of deduction). In what follows we will formulate only those additions to the system of general theory of deduction, which are necessary to obtain the corresponding sections of logic (theory of quantification, modal logic, etc.). We will call these additions proper parts of the given sections of logic.

CHAPTER EIGHT

SUBJECT-PREDICATE STRUCTURES

1. OBJECTS AND ATTRIBUTES

There are terms which are formed as follows:

1) one selects some object Π^1;

2) in the thus limited range of selection one selects some Π^2 different from Π^1; selection of Π^2 is impossible without that of Π^1; if the selection of Π^2 is carried out this means that the selection of Π^1 is carried out; which Π^1 is selected is of no importance for the selection of Π^2; the very selection of Π^1 provides the possibility of that of Π^2; this dependence of the selection of Π^2 on that of Π^1 occurs irrespective of assumptions on possibilities of the investigator (the dependence is assumed);

3) the Π^2 chosen in this way is designated by the term t^2 (we agree to consider an object of the type t^2 a term for Π^2).

We adopt the definitions:

D1. Object Π^2 is called the attribute and Π^1 the object.

D2. Terms which designate attributes are terms of attributes and those designating objects are terms of objects.

As we can see, the difference between attributes and objects is relative. We assume as given the ability to distinguish terms of attributes and terms of objects, in the case of simple terms.

We adopt the designations:

1) $S, S^1, S^2,\ldots$ are objects;

2) $P, P^1, P^2,\ldots$ are attributes;

3) $s, s^1, s^2,\ldots$ are terms of objects;

4) $P, P^1, P^2,\ldots$ are terms of attributes;

5) $s^*, s^{*1}, s^{*2},\ldots$ are terms: "object", "one (some) object", "another object", etc., respectively;

6) $P^*, P^{*1}, P^{*2},\ldots$ are terms: "attribute", "one attribute", ' another attribute", etc., respectively.

The terms introduced in 5) and 6) are, on the one hand, just special cases of terms mentioned in 3) and 4), and on the other they are most general terms of this type; i.e., for any s and P

$$s^* \rightharpoonup s, \quad s^{*i} \rightharpoonup s, \quad P^* \rightharpoonup P, \quad P^{*i} \rightharpoonup P.$$

From the definitions follows:

T1. If t^1 is a term of an object and t^2 is a term of an attribute, then for any t^1 and t^2

$$\sim (t^1 \rightharpoonup t^2) \cdot \sim (t^2 \rightharpoonup t^1).$$

T2. If $t^1 \rightharpoonup t^2$, $t^1 \rightleftharpoons t^2$ or $t^1 \equiv t^2$, then both t^1 and t^2 are terms of objects or terms of attributes.

The existence of attributes is determined relative to some method of selection (existential selection of the attribute). Only relative to objects must we here make the following reservation: an attribute exists if and only if there exists some object, the selection of which makes possible the existential selection of this attribute.

Thus attributes do not have an existence independent of objects.

From the definitions it also follows that the expression "attribute of attributes" is void of sense and the assertion of traditional logic, "The attribute of attributes is the attribute of a thing", is ambiguous, to say the least.

2. THE MOST ELEMENTARY SENTENCES

The most elementary sentences are formed from terms of objects and of attributes by uniting them with the aid of special operators. The latter will be called operators of predication (*D1*). A sentence will thus have the following parts: one and only one term of an object; one and only one term of an attribute; one and only one operator of predication (*D2*).

D3. The term of the object, found in the sentence referred to in *D2* will be called the subject of the sentence and the term of the attribute will be called the predicate.

The sentential elements named above are somehow (by some means or method) ordered in space or time. Subjects and predicates are independent in meaning. Though objects are selected in the establishment of the values of the predicates, this does not mean that one uses the subjects at hand.

We recall that we consider signs (including terms) as something physically invariable. It is therefore highly desirable that the reader divest himself of all the associations rooted in operating with ordinary language: subjects and predicates are to be taken as bereft of any grammatical properties, and the operators of predication as visible objects. In such a case the formation of sentences and terms is a set of operations, similar to those used to form mechanical structures from rigid elements.

We take the definitions:

D4. If s and P are, respectively, terms of object and attribute, then

$$s \leftarrow P,\ s \urcorner\!\leftarrow P,\ s? \leftarrow P$$

are sentences.

D5. The symbols

$$\leftarrow,\ \urcorner\!\leftarrow,\ ?\leftarrow$$

are, respectively, the operators of assertion, negation and indeterminacy.

D6. The symbols

$$\urcorner,\ ?$$

are, respectively, the signs of intrinsic negation and indeterminacy.

Instead of our symbol $s \leftarrow P$, the logician normally uses $P(s)$. So, the symbols $\urcorner P(s)$ and $?P(s)$ can be considered fully adequate to $s\urcorner\!\leftarrow P$ and $s? \leftarrow P$, respectively.

The operator of assertion "is read" as follows: "Object s has attribute P", "s is such, that P", "s is characterised by that which is P", etc. We will consider such expressions to be primitively clear, to be explained in examples. The following explanation is in place here: if one chooses S, then there is (known) some method of selection, through which in the given range of selection, and only in that range (i.e., without choosing other objects) P can be selected (the latter is experienced as the attribute of the former).

The operator of negation "reads" as follows: "s does not have the attribute P", "s is not such, that P", etc. It is also considered primitively clear. Its explanation: if we select S, then it is impossible to select P in this range of selection.

There are cases (anyway, we can assume that there are) where it is impossible to establish whether or not an object has a given attribute. This happens, for example, in the case of a change of objects. To such cases correspond sentences with the sign of indeterminacy.

Genetically, one would think that the ability to construct sentences with negations (and, even more, with signs of indeterminacy) presupposes the ability to construct sentences with assertions. But this must not be taken as saying that the construction of the former requires the use of the latter. All of these sentential forms can be simple in construction and independent one of the other (in particular they can be the results of direct observations).

Sentences of the type $s\alpha \leftarrow P$ (where α indicates the presence or absence of $\neg$ or ?) are constructed and used in a given (determined, known) context. They are localized in space and time, so that they can be called local. They serve for the description of isolated facts, events, etc., in certain circumstances and for the description of the results of observation and experiment. They are therefore sometimes called protocol. It would be erroneous to use the pronoun "this" in their formulation, for their localisation is something extrinsic and affects their use but is not fixed in their structure. Therefore, they can be used repeatedly and in different contexts.

For example, many people repeatedly say "a particle is negatively charged" but all such cases are samples of one and the same sentence and not different sentences.

These sentences cease to be local if the class of objects, about which they talk, is made up of only one object. Even here it does not cease in itself to be local, but only in conjunction with the knowledge that there is only one object designated by its subject. Most often, this supplementary knowledge is not explicitly formulated. But it is there: so that the sentence here is not simple but complex. For example, the sentence "This $s\alpha \leftarrow P$" is not simple: if the word "this" refers to some method of selection (e.g., gestures in the present case), then it is not a structural element of the sentence: it says only that $s\alpha \leftarrow P$ is local: if the "this" refers to the presence in the sentence of terms which describe the individual situation in question, then the sentence is not primitively simple.

It is not correct (although sometimes done) to consider $s \leftarrow P$ as a sentence on the inclusion of S in the class P. In the first place, a sentence on inclusion in a class is a sentence about two objects and has the form "t^1 is included in the class t^2". In the second place, S can be included in a class of objects which have the attribute P and not a class of attributes P, so that sentences ascribing an attribute to an object will all be presupposed as more simple.

It is not correct (but sometimes done) to consider $s \leftarrow P$ as the sentence "*s* is *P*". Sentences with "is" can be interpreted in different ways: as $t^1 \rightarrow t^2$, as $t^1 \rightleftharpoons t^2$, as $t^1 \equiv t^2$, as fixing that from the elements of some class one selects such a one, etc. In all cases, however, it is necessary that the terms be of the same order: both subjects or both predicates. All that has been said applies equally to sentences with negation and indeterminacy.

3. EXTRINSIC NEGATION

We take the following assertions as defining extrinsic negation:

A1. $\sim(s \leftarrow P) \equiv (s\urcorner \leftarrow P) : (s? \leftarrow P)$

A2. $\sim(s\urcorner \leftarrow P) \equiv (s \leftarrow P) : (s? \leftarrow P)$

A3. $\sim(s? \leftarrow P) \equiv (s \leftarrow P) : (s\urcorner \leftarrow P)$.

The sign of indeterminacy can be defined as derived from the assertion $(s? \leftarrow P) \equiv \; \sim(s \leftarrow P) \cdot \sim(s\urcorner \leftarrow P)$.

But, in order to do this, extrinsic negation has to be taken as primitively clear, and distinct from intrinsic negation.

The "classical" case is defined by the assertion:

A4. $\sim(s? \leftarrow P) \rightarrow (\sim(s \leftarrow P) \equiv (s\urcorner \leftarrow P))$

From *A1–A4* it is clear that the assertions

$\sim(s \leftarrow P) \dashv\vdash (s\urcorner \leftarrow P)$

$\sim(s\urcorner \leftarrow P) \dashv\vdash (s \leftarrow P)$

are valid only for classical objects and invalid for the non-classical, i.e., inadmissible for the general ("non-classical") case.

Already in the consideration of elementary sentences it becomes clear that one has to distinguish two types of negation which do not always coincide from a formal point of view. Their relationship is as follows: the sense of intrinsic negation is clear from its position in the sentence; the extrinsic negation has to be defined through the intrinsic one for each sentential structure. Such definitions, however, are not completely arbitrary. General negation has some independent sense which does not depend on the limited negation. In particular, its sense can be explained as follows: if someone asserts *X* and another (person) says "no", "it is not so", then this other asserts $\sim X$. It is formally defined by a system of assertions, like

$$\sim\sim X \vdash X, \; X \vdash \sim\sim X, \; X : \sim X, \; \sim(X \cdot \sim X)$$

etc., which we will fully elaborate below. Such types of definition must be taken into account when we define extrinsic negation by means of intrinsic negation for concrete sentences, i.e., there has to be consistency of these two paths for establishing the properties of extrinsic negation.

In the exposition which follows we will often talk about classical or non-classical cases, systems, etc. Non-classical cases will always mean that there are two different negations and indeterminacy; the classical will mean that indeterminacy is excluded and, therefore, the negations are not distinguished (or, they are identical). From this point of view the non-classical conception of logic appears as more general than the classical. The latter is in a sense a special case. This means that the usual conception of non-classical logic as a contraction of the classical has nothing in common with our terminology.

4. TERMS

From s, P and $\leftarrow$ we form the following terms:

1) $s\downarrow P$ is "s which has P", "s which P", "s having P", "s such that P", etc.;

2) $\downarrow(s\leftarrow P)$ is "that which is $s\leftarrow P$", "the fact that $s\leftarrow P$";

3) $P\downarrow s$ is "P present in s", "P of object s", "P such that it is present in s (which s has)".

Similarly, from s, P, $\neg\leftarrow$ and $?\leftarrow$ we form the terms:

1) $s\neg\downarrow P$, $s?\downarrow P$
2) $\downarrow(s\neg\leftarrow P)$, $\downarrow(s?\leftarrow P)$
3) $P\neg\downarrow s$, $P?\downarrow s$.

The term $P\neg\downarrow s$ "reads" as: "P such that it is not present in s (which s does not have)".

The sign $\sim$ is used to form the terms:

1) $s\sim\downarrow P$, $s\sim\neg\downarrow P$, $s\sim?\downarrow P$
2) $\downarrow\sim(s\leftarrow P)$, $\downarrow\sim(s\neg\leftarrow P)$, $\downarrow\sim(s?\leftarrow P)$
3) $P\sim\downarrow s$, $P\sim\neg\downarrow s$, $P\sim?\downarrow s$.

A special case of terms of this type occurs when for s and P we have the terms "object" (s^*) and "attribute" (P^*). For example, $s^*\downarrow P$ is "an object, having P". Not every such combination is practically useful.

Thus, $s^* \downarrow P^*$ coincides with s^* since any object has some attribute: $P \downarrow s^*$ coincides with P since every attribute is the attribute of some object.

To generalize we will use in what follows the letters α, β, γ, α^1, α^2,... before $\leftarrow$ and $\downarrow$ in the sense: each of them will singly indicate the presence or absence of $\urcorner$ or ?, and the meaning of their being distinguished in one assertion will be stated in every case.

What was said above about terms can be strengthened by the formal agreement:

D1. If s is a term of an object and P is a term of an attribute then

$$s\alpha \downarrow P,\ \downarrow(s\alpha \leftarrow P),\ P\alpha \downarrow s,\ s \sim \alpha \downarrow P,\ P \sim \alpha \downarrow s,\ \downarrow\sim(s\alpha \leftarrow P)$$

are terms; also

$$P\alpha \downarrow s,\ P \sim \alpha \downarrow s$$

are terms of attributes and the rest are terms of objects.

We will here indicate some of the properties of terms to be found in *D1* (without pretending to be complete). It is to be noted that within the framework of the same assertion identity of symbols indicates identity of the corresponding objects. We will also assume that for all sentences which figure in the same assertion there is an identity of time and place of selection of objects.

A1. $s\alpha \leftarrow (P\alpha \downarrow s)$
A2. $(s\alpha \downarrow P)\alpha \leftarrow P$
A3. $(s\alpha \downarrow P^1)\,\beta \downarrow P^2 \equiv (s\beta \downarrow P^2)\,\alpha \downarrow P^1$,

where α and β are different or identical in any of the combinations

A4. $s \rightharpoonup s\alpha \downarrow P$
A5. $s^* \downarrow (P \downarrow s) \rightharpoonup s,\ P^* \downarrow (s \downarrow P) \rightharpoonup P$
A6. $\sim s^1 \rightharpoonup (s^2 \urcorner \downarrow (P \downarrow s^1)),\ \sim s^1 \rightharpoonup (s^2 ? \downarrow (P \downarrow s^1))$
$\sim s^1 \rightharpoonup (s^2 \downarrow (P \urcorner \downarrow s^1)),\ \sim s^1 \rightharpoonup (s^2 \downarrow (P ? \downarrow s^1))$

A7. $(s\alpha \leftarrow P) \rightarrow (s^*\alpha \downarrow P \rightharpoonup s)$
A8. $(s\alpha \leftarrow P) \rightarrow (P^*\alpha \downarrow s \rightharpoonup P)$.

From *A1*II8 follows:

T1. $(s^1 \rightharpoonup s^2) \rightarrow (s^1\alpha \downarrow P \rightharpoonup s^2\alpha \downarrow P)$

$T2.\ (P^1 \rightharpoonup P^2) \rightarrow (s\alpha \downarrow P^1 \rightharpoonup s\alpha \downarrow P^2)$
$T3.\ (s^1 \rightharpoonup s^2) \cdot (P^1 \rightharpoonup P^2) \rightarrow (s^1\alpha \downarrow P^1 \rightharpoonup s^2\alpha \downarrow P^2).$

It should be clear from what has been said that the difference between subject and predicate is deeper than the simple difference of their positions in the sentence. For example, if we have the sentence $s\alpha \leftarrow P$, the simple transposition of s and P does not result in a sentence since an object of the type $P\alpha \leftarrow s$ is not a sentence. In order to get a sentence where P and s change places one has to construct the structure $(s^* \downarrow P)\,\alpha \leftarrow (P^* \downarrow s)$, $(s^*\alpha \downarrow P) \leftarrow (P^* \downarrow s)$, etc.

The following apply to predicates:

$A9.\ (P^1 \rightharpoonup P^2) \leftrightarrow ((s \downarrow P^2) \leftarrow P^1) \cdot ((s \urcorner \downarrow P^1) \urcorner \leftarrow P^2) \cdot ((s? \downarrow P^1)? \leftarrow P^2)$
$T4.\ (P^1 \rightharpoonup P^2) \cdot (s \leftarrow P^2) \rightarrow (s \leftarrow P^1)$
$(P^1 \rightharpoonup P^2) \cdot (s \urcorner \leftarrow P^1) \rightarrow (s \urcorner \leftarrow P^2)$
$(P^1 \rightharpoonup P^2) \cdot (s? \leftarrow P^1) \rightarrow (s? \leftarrow P^2).$

$A10.\ (s^1\alpha \leftarrow P)\,(s^2\beta \leftarrow P) \rightarrow (\sim s^1 \rightharpoonup s^2)$
$(s\alpha \leftarrow P^1)\,(s\beta \leftarrow P^2) \rightarrow (\sim P^1 \rightharpoonup P^2),$

where α and β are distinct.

$A11.\ ((s\alpha \leftarrow P^1) \rightarrow (s\beta \leftarrow P^2)) \rightarrow ((s\alpha \downarrow P^1)\,\beta \leftarrow P^2)$
$((s^1\alpha \leftarrow P) \rightarrow (s^2\beta \leftarrow P)) \rightarrow (s^2\beta \leftarrow (P\alpha \downarrow s^1)).$

$A12.$ $P \downarrow s$ is an individual term if and only if s is an individual term.

5. DEFINITIONS

The definitions of the terms of attributes (predicates) can have the form:

1) $s\alpha \leftarrow P^i \equiv s\beta \leftarrow P^k,$

where P^i is a newly introduced term, P^k and s are terms, s can be s^*, and α and β can be different or identical:

2) $s\alpha \leftarrow P^i \equiv (s\beta^1 \leftarrow P^1) \cdot \ldots \cdot (s\beta^n \leftarrow P^n),$

where $n \geqslant 2$, P^i is the term introduced, $P^1, \ldots, P^n$ and s are terms, α, $\beta^1, \ldots, \beta^n$ are pairwise identical or different in some combination;

3) $s\alpha \leftarrow P^i \equiv (s\beta^1 \leftarrow P^1) : \ldots : (s\beta^n \leftarrow P^n);$
4) $s\alpha \leftarrow P^i \equiv X^1 : \ldots : X^m,$

where $m \geqslant 2$, $X^1, \ldots, X^m$ are sentences of the form $(s\beta^{i1} \leftarrow P^{i1}) \cdot \ldots \cdot (s\beta^{ik} \leftarrow P^{ik})$, where $k \geqslant 1$.

Similar definitions are also possible for subjects. The following definitions are special cases of these:

D1. $(s\alpha \leftarrow (\cdot P^1, \ldots, P^n)) \equiv (s\alpha \leftarrow P^1) \cdot \ldots \cdot (s\alpha \leftarrow P^n)$
D2. $(s\alpha \leftarrow (\vee P^1, \ldots, P^n)) \equiv (s\alpha \leftarrow P^1) \vee \ldots \vee (s\alpha \leftarrow P^n)$
D3. $((\cdot s^1, \ldots, s^n)\, \alpha \leftarrow P) \equiv (s^1\alpha \leftarrow P) \cdot \ldots \cdot (s^n\alpha \leftarrow P)$
D4. $((\vee s^1, \ldots, s^n)\, \alpha \leftarrow P) \equiv (s^1\alpha \leftarrow P) \vee \ldots \vee (s^n\alpha \leftarrow P)$
D5. $(s\alpha \leftarrow \tilde{P}) \equiv \sim (s\alpha \leftarrow P)$.

The following holds for terms introduced according to the first schema of definitions:

A1. $((s \leftarrow P^i) \equiv (s \leftarrow P^k)) \rightarrow (P^i \rightleftharpoons P^k)$.

T1. $((s \leftarrow P^i) \equiv (s \leftarrow P^k)) \rightarrow (s \downarrow P^i \rightleftharpoons s \downarrow P^k)$.

In defining the terms of attributes (predicates) one defines not the signs P in themselves but the sentences $s \leftarrow P$ or $s^* \leftarrow P$ which contain them. It is important to keep this in mind when explaining the sense of terms such as "true", "false", "exists", etc. It is foolish to ask "What is truth?", "What is existence?", etc. One can ask questions only on the sense of expressions like "[X] is true", "[X] is false", "s exists", etc., and this only if certain considerations relative to X and s (structure, type of object, etc.) are taken into account.

Terms for objects are defined through

$$s^1 = Df.\, s^2 \downarrow P,$$

where s^2 can be s^* and P is a predicate in the above-mentioned generalised sense. In another form: "If $s^2 \leftarrow P$, then s^2 will be called s^1". This definition has the property:

A2. $(s^1 = Df.\, s^2 \downarrow P) \rightarrow (s^1 \leftarrow P)$

Let D^i be the definition of s^1 such that "If D^i, then $s^1 \leftarrow P$". Then:

A3. $(s^2 \urcorner \leftarrow P) \rightarrow \sim (s^1 \rightharpoonup s^2)$
$(s^2 \quad ? \leftarrow P) \rightarrow \sim (s^1 \rightharpoonup s^2)$.

A2 makes it possible to construct definitions which differ from those constructed according to the above schema (they are called explicit) in that they have the form of axioms: in such a case we simply take an assertion of the type

$$s^1\alpha^1 \leftarrow P^1, \ldots, s^1\alpha^n \leftarrow P^n,$$

implicitly presupposing that

$$s^1 = Df. s^2 \downarrow (\alpha^1 P^1 \cdot \ldots \cdot \alpha^n P^n).$$

This way of writing definitions is called implicit definition.

From what has been said it should be clear why definitions figure as premisses in judgements (in inferences).

6. RULES OF SUBSTITUTION OF TERMS

Definitions according to the above schema are one-act, i.e., they are agreements relative to the sense of fully defined, separate terms. The rules of the substitution of terms are certain standard rules for obtaining new terms from old ones, which hold for some classes of terms: they can serve repeatedly (for different sets of the terms in question). They are used to obtain new terms of attributes and act in the following way: 1) there are rules for obtaining a certain type of new predicates from the given predicates; 2) according to these rules the sentence $s \leftarrow P$ is obtained from the sentences $s \leftarrow P^1, \ldots, s \leftarrow P^n$. For example: from the sentences "s weighs a^1 kilos",..., "s weighs a^n kilos" one gets the sentence "s weighs an average of b kilos"; from "s travelled a distance of a kilometers" and "s spent b hours doing it" one gets "s has the speed c km/h".

We should note the following: the term P is not defined here through $P^1, \ldots, P^n$; it has meaning independently of them. And the sentence $s \leftarrow P$ has meaning independently of $s \leftarrow P^1, \ldots, s \leftarrow P^n$. The rules we are talking about here, first, provide a definition of a term P such that $P^k \rightarrow P$, through terms $P^{i1}, \ldots, P^{in}$ such that $P^{i1} \rightarrow P^1, \ldots, P^{in} \rightarrow P^n$; secondly, they make it possible to find a definition of P from the value-range P^k for $P^1, \ldots, P^n$ from the value-range $P^{i1}, \ldots, P^{in}$ (i.e., P^k is a function from arguments $P^{i1}, \ldots, P^{in}$).

7. INDIVIDUALIZATION OF TERMS

D1. We will call individualization of the term s construction of a term $s\downarrow P$ such that it is individual. P is here an individualizing predicate. A common method of individualization is indication of the place and time of existence of the object. But the individualizing portion of the terms is usually distinguished from the base (i.e., from s), brought from outside as something general to many terms used in the context in question, or generally presupposed in implicit form.

The sign "this" is the method of individualization. It indicates that that which the investigator, operating with the term t, selects as an individual object is designated by it. It is clear that $t \rightarrow$ "this t".

8. SENTENCES ON *n*-TUPLES OF OBJECTS

Science has to do not only with single objects but also with pairs, triples and, in general, with n-tuples of objects. This will be represented as

$$(\Pi^1,\ldots,\Pi^n),$$

and the corresponding terms as

$$(t^1,\ldots,t^n).$$

We assume the following definitions and expansions of previously accepted agreements.

D1. Object Π is an n-tuple of objects (Π); objects $\Pi^1,\ldots,\Pi^n$ ($n \geqslant 2$), for any pair of which Π^i and $\Pi^k \sim (t^i \rightarrow t^k)$ is met, form n-tuples of objects $(\Pi^i,\ldots,\Pi^n)$; objects $\Pi^i,\ldots,\Pi^n$ here are elements of the n-tuple.

D2. Every n-tuple of objects is an object.

D3. To select the n-tuple of objects $(\Pi^1,\ldots,\Pi^n)$ means to select all its elements $\Pi^1,\ldots,\Pi^n$ in the indicated order.

D4. If $t^i,\ldots,t^n$ are terms, then $(t^1,\ldots,t^n)$ is an n-tuple of terms; the latter is, in turn, a term.

If $\Pi^1,\ldots,\Pi^n$ are distinguished only as one, other, etc. Π, then the n-tuple of the terms $(t^1,\ldots,t^n)$ can take the form "two (a pair of) t", "three t",..., "nt". Terms like "n-tuple of objects", "n-tuple of attributes" are also possible.

The relationship between terms found in n-tuples of terms and the n-tuple itself is determined by:

A1. $(t^i \rightharpoonup t^k) \rightarrow ((t^1, \ldots, t^n) \rightharpoonup (t^1, \ldots, t^n)(t^i/t^k))$,
where t^i is any one of $t^1, \ldots, t^n$.

A2. $\sim(t^i \rightharpoonup (t^1, \ldots, t^n))$, $\sim((t^1, \ldots, t^n) \rightharpoonup t^i)$.

D5. Depending on the number of objects which have to be selected for the selection of an attribute in the construction of a term designating it, we will speak about one-place, two-place and, in general, about *n*-place attributes and, correspondingly, of terms of attributes.

D6. A one-place attribute is called a property; two and more place attributes are called relations. A relation is an attribute of an *n*-tuple of two or more objects.

D7. An *n*-tuple of objects is an object.

D8. The rule of formation of sentences with *n*-place predicates (or sentences about *n*-tuples of objects):

1) one sentence is formed of two and only two terms, one of which is the term of an object and the other the term of a predicate, and of the operator;

2) if the term of the attribute is *n*-place then in the *n*-tuple of terms, being the term of the object, there has to be a corresponding number (*n*) of terms.

D9. The term of the object, referred to in *D8*, is the subject of the sentence and the term of the attribute is the predicate.

We will represent such sentences with the symbols

$$(s^1, \ldots, s^n)\,\alpha \leftarrow P \quad \text{or} \quad \alpha P(s^1, \ldots, s^n).$$

The definition of the simple sentence can be generalized as follows:

D10. If a is an *n*-tuple of subjects, and Q the corresponding *n*-place predicate, then $a\alpha \leftarrow Q$ (or $\alpha\, Q(a)$) is a structurally simple sentence.

Since sentences are constructed according to the rules of a given language, regardless of the abstractions and assumptions of logic, strict localization of the subject and predicate can be attained only through the following operations (let X be a sentence, made up of subject, predicate and operator and not containing any other parts):

1) from X are taken out all $s^1, \ldots, s^n$ and from them is formed an *n*-tuple of terms as the subject of the given sentence: we assume that terms of objects are eliminated and all are taken out (such is the practice);

2) in the empty places are inscribed terms $s^{*1}, \ldots, s^{*n}$ so that the indices of s^*, fixing the indices of the order of selection of the objects, correspond

to the sequence of $s^1, \dots, s^n$ in the subject; the sentence obtained is

$$X(s^1, \dots, s^n / s^{*1}, \dots, s^{*n});$$

3) sentence X can now be written in the form "$(s^1, \dots, s^n)$ has this attribute that $X(s^1, \dots, s^n / s^{*1}, \dots, s^{*n})$"; if X^* is such an inscription, then

$$X \equiv X^*.$$

For example, the sentence "s^1 is found between s^2 and s^3" is an object of the type "(s^1, s^2, s^3) is such that the first is found between the second and third".

This operation is not something developed especially for logic and for logic alone. The fact is that the predicate in X has meaning independent of the object and it can be established in a way which describes this operation in a generalized form.

All that has been said about sentences with one-place predicates can be extended to sentences with many-place predicates considered as a special case, since the n-tuple of terms forming the subject and the many-place predicate are respectively subject and predicate which can be regarded as something undifferentiated and elementary. But since the subject here can include more than one term, there are additional possibilities. Below we will limit ourselves to the most essential of them.

9. TRANSFORMATION RULES AND TERMS

Let X be $(s^1, \dots, s^n)\ \alpha \leftarrow P$. It can take the form $(1 \leqslant m \leqslant n)$:

1) "s^i has an attribute such that X"
2) "s^i has an attribute such that $X(s^i/s^*)$"
3) "$(s^{i1}, \dots, s^{im})$ has an attribute such that X"
4) "$(s^{i1}, \dots, s^{im})$ has an attribute such that $X(s^{i1}, \dots, s^{im} / s^{*1}, \dots, s^{*m})$".

The symbols

$$PX,\ PX(s^i/s^*),\ PX(s^{i1}, \dots, s^{im} / s^{*1}, \dots, s^{*m})$$

designate the predicates obtained in this way. Since this subdividing is only a change in our point of view toward X, the truth-value of X does not change. So that

$$A1.\ X \leftrightarrow (s^i \leftarrow PX^0),\ X \leftrightarrow ((s^{i1}, \dots, s^{im}) \leftarrow PX^0),$$

where X^0 can be X and $X(s^i/s^*)$ or $X(s^{i1},\dots, s^{im}/s^{*1},\dots, s^{*m})$. We also assume

$$
\begin{aligned}
&A2.\ ((s^1, s^2, s^3) \leftarrow PX^0) \leftrightarrow ((s^1, (s^2, s^3)) \leftarrow PX^0)\\
&T1.\ \sim (s^i \leftarrow PX^0) \leftrightarrow (s^i \leftarrow P \sim X^0)\\
&\quad\ \sim ((s^{i1}, \dots, s^{im}) \leftarrow PX^0) \leftrightarrow ((s^{i1}, \dots, s^{im}) \leftarrow P \sim X^0).\\
&A3.\ s \downarrow P(s\alpha \leftarrow Q) \equiv s\alpha \downarrow Q.
\end{aligned}
$$

These assertions are transformation rules of X. The consequence:

$$
\begin{aligned}
&T2.\ (s^{i1} \leftarrow PX^0) \leftrightarrow (s^{i2} \leftarrow PX^0)\\
&\quad\ ((s^{i1}, \dots, s^{im}) \leftarrow PX^0) \leftrightarrow ((s^{k1}, \dots, s^{kl}) \leftarrow PX^0),
\end{aligned}
$$

where $s^{i1}, s^{i2},\dots, s^{im}, s^{k1},\dots, s^{kl}$ are terms from the set $s^1,\dots, s^n$.

From X are formed the terms:

1) PX^0 are terms of attributes;
2) $s^i \downarrow PX^0, (s^{i1}, \dots, s^{im}) \downarrow PX^0, s^* \downarrow PX(s^i/s^*)$,
$(s^{*1},\dots, s^{*m}) \downarrow PX(s^{i1},\dots, s^{im}/s^{*1},\dots, s^{*m})$ are terms of objects;
3) $\downarrow(X), \downarrow(X(s^i/s^*)), \downarrow(X(s^{i1}, \dots, s^{im}/s^{*1}, \dots, s^{*m}))$,
$\downarrow(s^i \leftarrow PX^0), \downarrow((s^{i1},\dots, s^{im}) \leftarrow PX^0)$ are terms of objects.

The transformation rules and the means of forming terms, formulated above, can be generalized by considering X to be any sentence. For example, the sentence $(s^1 \leftarrow P^1)\cdot(s^2 \leftarrow P^2)$ can be presented in the form $(s^1, s^2) \leftarrow P((s^{*1} \leftarrow P^1)\cdot(s^{*2} \leftarrow P^2))$ with all the consequences which result therefrom.

10. DEFINITIONS

Agreements of the type

$$
\begin{aligned}
&(s^i \leftarrow P^i) \equiv X, (s^* \leftarrow P^i) \equiv X(s^i/s^*)\\
&((s^{i1}, \dots, s^{im}) \leftarrow P^i) \equiv X, ((s^{*1}, \dots, s^{*m}) \leftarrow P^i) \equiv\\
&\equiv X(s^{i1}, \dots, s^{im}/s^{*1}, \dots, s^{*m}),
\end{aligned}
$$

serving as definitions of predicates P^i, are possible.

11. EXISTENTIAL PREDICATE

Instead of the word "exists" in the sense defined above, we will use the symbol E. We will consider it a simple predicate. Sentences with this predicate will have the form

$$s\alpha \leftarrow E \quad \text{or} \quad \alpha E(s).$$

There are cases where it cannot be established if the object exists. This means that consideration of sentences like $s? \leftarrow E$ is fully justified.

All that was said above about predicates applies to E. But it also has a series of properties as a special logical (studied in logic) predicate. For example,

$$\begin{aligned}
&A1.\ (s^1 \rightharpoonup s^2)(s^2 \leftarrow E) \rightarrow (s^1 \leftarrow E)\\
&(s^1 \rightharpoonup s^2)(s^1 \urcorner \leftarrow E) \rightarrow (s^2 \urcorner \leftarrow E)\\
&(s^1 \rightharpoonup s^2) \sim (s^1 \leftarrow E) \rightarrow \sim (s^2 \leftarrow E)\\
&A2.\ (s \leftarrow E)((s \downarrow P) \urcorner \leftarrow E) \rightarrow ((s \urcorner \downarrow P) \leftarrow E)\\
&(s \leftarrow E)((s \downarrow P)? \leftarrow E) \rightarrow ((s? \downarrow P) \leftarrow E)\\
&(s \leftarrow E) \sim ((s \downarrow P) \leftarrow E) \rightarrow ((s \sim \downarrow P) \leftarrow E),
\end{aligned}$$

$$\begin{aligned}
&T1.\ ((s\alpha \downarrow P) \leftarrow E) \rightarrow (s \leftarrow E)\\
&(s \urcorner \leftarrow E) \rightarrow ((s\alpha \downarrow P) \urcorner \leftarrow E),
\end{aligned}$$

where E does not occur in P.

$$\begin{aligned}
&A3.\ ((s^1, \ldots, s^n) \leftarrow E) \equiv (s^1 \leftarrow E) \cdot \ldots \cdot (s^n \leftarrow E)\\
&((s^1, \ldots, s^n) \urcorner \leftarrow E) \equiv (s^1 \urcorner \leftarrow E) \vee \ldots \vee (s^n \urcorner \leftarrow E)\\
&(s^1 \rightharpoonup s^2) \sim (s^1 \leftarrow E) \rightarrow \sim (s^2 \leftarrow E)\\
& \sim (s^1 \urcorner \leftarrow E) \cdot \ldots \cdot \sim (s^n \urcorner \leftarrow E),\\
&T2.\ (a \leftarrow E)((a \downarrow PX) \urcorner \leftarrow E) \rightarrow ((a \downarrow P \sim X) \leftarrow E)\\
&(s \urcorner \leftarrow E) \rightarrow ((s \downarrow X) \urcorner \leftarrow E)\\
&((s^1 \downarrow X) \leftarrow E) \rightarrow ((s^2 \downarrow X) \leftarrow E),
\end{aligned}$$

where E does not occur in X and s^1 and s^2 occur in X.

12. TWO TYPES OF OBJECTS AND ATTRIBUTES

Objects are divided into two groups as follows:

1) in the first group belong objects for which the possibility that $(s? \leftarrow E)$ is excluded;

2) in the second are objects for which such a possibility is not excluded

Similarly, attributes are divided into two groups as follows:

1) the first group includes attributes for which the possibitliy that $(s \leftarrow E) \cdot (s? \leftarrow P)$ is excluded;

2) the second group includes attributes for which such a possibility is not excluded.

D1. We will call objects and attributes of the first group classical and those of the second group non-classical.

Logical assertions about non-classical objects and attributes are more general than those about classical objects and attributes: the latter are obtained from the former as special cases through elimination of all sentences (and terms) with the sign of indeterminacy.

13. TRUTH-VALUES

Strictly speaking, the symbols $X \leftarrow v^i$ and $\sim(X \leftarrow v^i)$, used above, should be replaced by something like: $[X] \leftarrow v^i$ and $[X] \neg \leftarrow v^i$. Since indeterminacy is excluded for truth-values, i.e., $[X]? \leftarrow v^i$, one can write the assertion

$$\sim([X] \leftarrow v^i) \equiv ([X] \neg \leftarrow v^i)$$

or

$$\sim([X]? \leftarrow v^i).$$

D1. $[s\alpha \leftarrow E] \leftarrow v^1 \equiv (s\alpha \leftarrow E)$

D2. $[s \leftarrow E] \leftarrow v^4 \equiv [s \neg \leftarrow E] \leftarrow v^1$
$[s \neg \leftarrow E] \leftarrow v^4 \equiv [s \leftarrow E] \leftarrow v^1$
$[s? \leftarrow E] \leftarrow v^4 \equiv ([s \leftarrow E] \leftarrow v^1) : ([s \neg \leftarrow E] \leftarrow v^1)$

D3. $[s \leftarrow E] \leftarrow v^2 \equiv [s? \leftarrow E] \leftarrow v^1$
$[s \neg \leftarrow E] \leftarrow v^2 \equiv [s? \leftarrow E] \leftarrow v^1$

D4. $[s\alpha \leftarrow P] \leftarrow v^1$ in confrontation with an existing object, designated by the term $s\alpha \downarrow P$.

D5. $[s \leftarrow P] \leftarrow v^4 \equiv [s \neg \leftarrow P] \leftarrow v^1$
$[s \neg \leftarrow P] \leftarrow v^4 \equiv [s \leftarrow P] \leftarrow v^1$
$[s? \leftarrow P] \leftarrow v^4 \equiv ([s \leftarrow P] \leftarrow v^1) : ([s \neg \leftarrow P] \leftarrow v^1)$

D6. $[s \leftarrow P] \leftarrow v^2 \equiv [s? \leftarrow P] \leftarrow v^1$
$[s \neg \leftarrow P] \leftarrow v^2 \equiv [s? \leftarrow P] \leftarrow v^1$

D7. $[s\alpha \leftarrow P] \leftarrow v^3 \equiv ([s \neg \leftarrow E] \leftarrow v^1) : ([s? \leftarrow E] \leftarrow v^1)$

In definitions *D5–D7* the predicate E does not occur in $.P$

T1. In accordance with the accepted definitions it can be established that the following sentences are, respectively, tautologies:

1) $(s \leftarrow E):(s\neg \leftarrow E):(s? \leftarrow E)$
2) $\sim (s \leftarrow E):((s \leftarrow P):(s\neg \leftarrow P):(s? \leftarrow P))$
3) $\sim ((s \leftarrow P)\cdot(s\neg \leftarrow P)),\ \sim ((s \leftarrow P)\cdot(s? \leftarrow P))$
$\sim ((s\neg \leftarrow P)\cdot(s? \leftarrow P)),$
4) $\sim ((s \leftarrow E)\cdot(s\neg \leftarrow E)),\ \sim ((s \leftarrow E)\cdot(s? \leftarrow E)),$
$\sim ((s\neg \leftarrow E)\cdot(s? \leftarrow E))$
5) $\sim (\sim (s \leftarrow E)\cdot(s\alpha \leftarrow P)).$

T2. If the objects and attributes are classical, then the sentences

$$(s \leftarrow E):(s\neg \leftarrow E)$$
$$\sim (s \leftarrow E):((s \leftarrow P):(s\neg \leftarrow P))$$

are tautologies; if both object and attribute are classical, then

$$(s\neg \leftarrow E):((s \leftarrow P):(s\neg \leftarrow P))$$

is a tautology. But in the general case these sentences will not be tautologies. We also note that

$$(s \leftarrow P):(s\neg \leftarrow P):(s? \leftarrow P)$$
$$(s \leftarrow P):(s\neg \leftarrow P)$$

are not always true: they can be unprovable in the case $\sim (s \leftarrow E)$.

T3. The sentences

$$(s\alpha \leftarrow P)\cdot(s\beta \leftarrow P)$$
$$(s \leftarrow P)\cdot(s\neg \leftarrow P)\cdot(s? \leftarrow P)$$
$$(s\alpha \leftarrow E)\cdot(s\beta \leftarrow E),$$

where α and β are different, are unsatisfiable. But they are not contradictions (in our sense) since they can have the value v^2 and (for the first two) v^3. But the sentences

$$(s \leftarrow E)\cdot(s\neg \leftarrow E)\cdot(s? \leftarrow E)$$
$$\sim (s \leftarrow E)\cdot(s\alpha \leftarrow P)$$

are contradictions.

It can also be established that

T4. $(s\alpha \leftarrow P) \vdash \sim (s\beta \leftarrow P)\cdot \sim (s\gamma \leftarrow P),$

where α, β and γ are such that if one of them designates the lack of $\neg$

and?, then of the two others one indicates $\neg$ and the other indicates?.

T5. $\sim(s\alpha \leftarrow P)\cdot \sim(s\beta \leftarrow P) \vdash (s\gamma \leftarrow P)$
T6. $(s\alpha \leftarrow E) \vdash \sim(s\beta \leftarrow E)\cdot \sim(s\gamma \leftarrow E)$
T7. $\sim(s\alpha \leftarrow E)\cdot \sim(s\beta \leftarrow E) \vdash (s\gamma \leftarrow E)$.

14. THEORY OF PREDICATION

The theory of predication S^p is obtained through the following additions to the systems S^i (or S_i).

Designations:

1) $s, s^1, s^2, \ldots$ are subject variables (individual variables);

2) $P, P^1, P^2, \ldots$ are predicate variables.

D1. The group of subject variables:

1) a subject variable is a group of subject variables;

2) if $a^1, \ldots, a^n$ $(n \geqslant 2)$ are groups of subjects variables, then $(a^1, \ldots, a^n)$ is a group of subject variables;

3) something is a group of subject variables only by virtue of 1 and 2.

D2. The elementary subject-predicate formula: if Q is a predicate variable and a is a group of subject variables, then $a \leftarrow Q$, $a\neg \leftarrow Q$ and $a? \leftarrow Q$ are elementary subject-predicate formulae (or $Q(a)$, $\neg Q(a)$ and $?Q(a)$).

D3. SP-formula:

1) an elementary subject-predicate formula is an SP-formula;

2) if $x^1, \ldots, x^n$ are SP-formulae, then $\sim x^1$, $x^1 \cdot \ldots \cdot x^n$ and $x^1 : \ldots : x^n$ (correspondingly $x^1 \vee \ldots \vee x^n$) are SP-formulae;

3) if a is a group of subject variables and X is an SP-formula, then $a \leftarrow PX$ (or $PX(a)$) is an SP-formula.

D4. Occurrence:

1) s occurs in $s\alpha \leftarrow P$;

2) P occurs in $s\alpha \leftarrow P$ and $(s^1, \ldots, s^n)\alpha \leftarrow P$;

3) s^i occurs in $(s^1, \ldots, s^n)$ $(n \geqslant 2;\ i = 1, \ldots, n)$;

4) s^i occurs in $(s^1, \ldots, s^n)\alpha \leftarrow P$.

Axioms *AI*:

1. $\sim(s \leftarrow P) \dashv\vdash (s\neg \leftarrow P) \vee (s? \leftarrow P)$
2. $\sim(s\neg \leftarrow P) \dashv\vdash (s \leftarrow P) \vee (s? \leftarrow P)$
3. $\sim(s? \leftarrow P) \dashv\vdash (s \leftarrow P) \vee (s\neg \leftarrow P)$.

Axiomatic schema *AII*:

$$(a \leftarrow PX) \dashv\vdash X,$$

where a is a group of subject variables and X is an SP-formula.

Rules of inference:

R1. Substitution of formulae for sentential variables.

R2. Substitution of groups of subject variables for subject variables.

R3. Substitution of predicate variables for predicate variables.

Interpretation: 1) if one of $a \leftarrow Q$ and $a \neg \leftarrow Q$ takes the value v^1, then the other takes the value nv^1; if one of them takes the value nv^1, then the value of the other does not depend on the first; 2) $a? \leftarrow Q$ is equivalent to $\sim (a \leftarrow Q) \sim (a \neg \leftarrow Q)$; 3) $a \leftarrow PX$ is equivalent to X.

T1. In such an interpretation all formulae provable in S^p are tautologies.

Suitable for $a \neg \leftarrow Q$ is also the interpretation $\sim (a \leftarrow Q) \cdot \alpha$, where α is a sentential variable which does not occur in a formula in which $a \neg \leftarrow Q$ occurs.

D4. We will call the predicates P^1 and P^2 logically interchangeable if and only if

$$\begin{aligned} &(s \leftarrow P^1) \dashv\vdash (\sim s \neg \leftarrow P^2) \\ &(s \neg \leftarrow P^1) \dashv\vdash (\sim s \leftarrow P^2) \\ &(s? \leftarrow P^1) \dashv\vdash (\sim s? \leftarrow P^2) \end{aligned}$$

are valid.

D5. The predicate P^1 is deductively stronger than P^2 if and only if

$$\begin{aligned} &(s \leftarrow P^1) \vdash (s \leftarrow P^2) \\ &\sim ((s \leftarrow P^2) \vdash (s \leftarrow P^1)) \end{aligned}$$

T2. In S^p

$$\begin{aligned} &(s\alpha \leftarrow P) \dashv\vdash \sim (s\beta \leftarrow P) \sim (s\gamma \leftarrow P) \\ &\vdash \sim ((s\alpha \leftarrow P)(s\beta \leftarrow P)) \\ &\vdash (s \leftarrow P) : (s \neg \leftarrow P) : (s? \leftarrow P) \\ &\sim (s\alpha \leftarrow P) \dashv\vdash (s\beta \leftarrow P) : (s\gamma \leftarrow P) \end{aligned}$$

are provable.

CHAPTER NINE

EMPIRICAL AND ABSTRACT OBJECTS

1. EMPIRICAL OBJECTS

D1. Real empirical objects are objects which are reflected by the investigator through his natural (sense-) apparatus of reflection, which act on this apparatus (are sensed, perceived), and are observed by the investigator. Observation can be carried out with the help of instruments which multiply the capacity of sense reflection and can be ordered in space and time as a series of such acts of reflection. The problem of the existence of such objects is (ultimately) solved by the possibility of their being observed (by the investigator in question or by others who are trustworthy). If the investigator decides on the existence of such objects in the past or in a place where he cannot observe them, an implicit assumption is made, to wit if the investigator could transpose himself in space or time to a position relative to this object, the latter would be accessible to observation.

Real empirical objects are not eternal (they come to be and pass away); they are changeable (they lose one attribute and acquire another); they exist in a definite milieu, in a determined space and time; they are the effects of causes and are themselves causes of effects; they have an infinite number of different attributes, etc. Sentences about them (having the form $s\alpha \leftarrow E$ and $s\alpha \leftarrow P$), their extrinsic negations and combinations by means of signs "and", "or", etc., can have different truth-values depending on time and place (one and the same sentence can be true in one time or place and false in another).

D2. Hypothetical empirical objects are objects with the following characteristics. In themselves they are not observed nor are they in principle observable, because of the relation between the properties and means of the investigator and of these objects. What is observed is their influence on other observable objects, which are considered as their attributes. The existence of these objects is assumed for precise goals. These objects (like the real ones) are assumed as coming-to-be and passing away, as changeable, etc. The basic principles of their being assumed are:

1) the logical consistency of sentences about them; the lack of contradiction between these sentences and the theses of the science in question;

2) the goal for which they are assumed is attained. Elementary particles in physics are examples of hypothetical empirical objects.

Logical consistency and inconsistency is here usually understood as: X is logically inconsistent if there is a Y such that $X \vdash Y \cdot \sim Y$, and consistent if there is no such Y (i.e., $\sim (X \vdash Y \cdot \sim Y)$, where Y is any sentence).

D3. Real empirical objects and hypothetical empirical objects are empirical objects.

2. ABSTRACT OBJECTS

In an act of knowledge the investigator can decide not to take into consideration some attributes of the objects (exclusive-negative abstraction) or to consider only certain attributes of the objects (selective-positive abstraction). This decision can be effected in single cases by selecting an object-range in which the objects being investigated really do not have the attributes in question, or by artificially creating such a range. In these cases the objects being investigated remain empirical, being selected only in certain cases for observation.

Things are different if one decides to abstract from attributes of the objects, without which empirical objects in general or the objects of the range in question cannot exist. The same is true of selective abstraction since the decision to consider only certain attributes means not considering the others. For example, the investigator decides not to consider the extension and form of physical bodies while studying their motion, taking these bodies as not having spatial extension (they are "material points").

D1. Effecting this decision involves the assumption of special objects, which are called abstract (in any case, we will so call them). By the very way they are assumed, these objects do not exist empirically. Investigation of them will not be a matter of observation. Below we will clarify a series of concepts relative to abstract objects.

Abstract objects are introduced into science in the following way. The initial (or primitive) abstract objects are introduced by the usual definitions with additional specifications on the exclusion of attributes, as described above. The essence of these definitions can be presented schematically.

Schema I: object s will be the term of an object such that

$$s = Df.(s^* \downarrow (P^1 \cdot \ldots \cdot P^n \cdot \neg P_1 \cdot \ldots \cdot \neg P_m)) \quad (n \geqslant 1, m \geqslant 1).$$

One here enumerates the attributes which are present in the object introduced (this is not always done explicitly) and which are not present (are excluded); the excluded attributes are such that the empirical objects (in general or in the given domain of science) cannot exist without them, i.e., if s^i is an empirical object of the domain being investigated, then

$$(s^i \downarrow (P^1 \cdot \ldots \cdot P^n)) \leftarrow (P_1 \cdot \ldots \cdot P_m), \sim (s^i \downarrow (P^1 \cdot \ldots \cdot P^n) \neg \leftarrow \\ \leftarrow (P_1 \vee \ldots \vee P_m)).$$

In another form, a definition according to schema I can take the form of a definition of the object itself, according to the principle "s is an object such that...". A definition can also have the form of a system of axioms with the primitive term s:

$$s \leftarrow P^1, \ldots, s \leftarrow P^n, s \neg \leftarrow P_1, \ldots, s \neg \leftarrow P_m.$$

Schema II: object s will be a term of an object such that

$$s = Df. s^* \downarrow (P^1 \cdot \ldots \cdot P^n),$$

where $n \geqslant 1$, and if from this convention and the other definitions and assertions of the science in question it does not logically follow that $s \leftarrow P^k$, then $s \neg \leftarrow P^k$; P^k is here a necessary attribute of empirical objects, i.e.,

$$(s^i \downarrow (P^1 \cdot \ldots \cdot P^n)) \leftarrow P^k, \sim (s^i \downarrow (P^1 \cdot \ldots \cdot P^n) \neg \leftarrow P^k).$$

This definition can also be written in the form of a system of axioms

$$s \leftarrow P^1, \ldots, s \leftarrow P^n.$$

D2. Objects which are designated by terms which are defined according to these schemata are called primitive abstract objects.

We should note that in the definitions of the terms of the primitive abstract objects there are no terms of abstract objects other than the newly introduced terms.

D3. A primitive abstract object exists if and only if one observes the rules of definition for the introduction of its term, and if from the definitions of its terms and of the other definitions and assertions of the science in question do not follow logical contradictions, i.e., provided these other definitions and assertions are consistent.

From *D2* and *D3* follows:

T1. A primitive abstract object either exists or does not exist and indeterminacy is excluded, i.e.,

$$(sa \leftarrow E):(sa \urcorner \leftarrow E),$$

where $s\alpha$ is a primitive abstract object.

Since one tries to formulate definitions of primitive abstract objects so as to meet *D3*, these objects will always be assumed as existing (points, lines, numbers, etc., are assumed to be given).

T2. Sentences on primitive abstract objects are universal.

D4. Derivative abstract objects are objects whose terms are defined with the help of primitive abstract objects.

T3. The question concerning the existence of derivative abstract objects is decided by means of reasonings, i.e., by inference of the corresponding assertions or their negations from the definitions of the primitive abstract objects or by establishing the impossibility of such inferences. There are, therefore, three possible results here: proof of existence; proof of non-existence; establishment of the undecidability of the problem of existence.

T4. The attributes of derivative abstract objects are also explicated through reasonings. Three results are possible here, too.

D5. Primitive and derivative abstract objects are abstract objects.

A1. If $(s^1 \rightharpoonup s^2)$ and $s^1 (s^2)$ are empirical (abstract) objects, then $s^2 (s^1)$ is an empirical (abstract) object.

3. INTERPRETATION

D1. Interpretation of an abstract object s^1 consists in the following:

1) object s^2 is correlated with the abstract object s^1, i.e., one establishes

$$s^2 \Leftarrow s^1;$$

2) s^2 is selected in such a way that for any X the following is satisfied

$$(s^1 \leftarrow PX) \rightarrow (s^2 \leftarrow PX(s^1/s^2)).$$

D2. An abstract object which has an interpretation is called a real abstract object; without interpretation it is called hypothetical. The latter are introduced for purposes of deduction.

We should note that interpretation of the term of an abstract object and selection of a term from the value-range of a term are different operations. They are often confused since in both cases there is a reasoning from s^1 to s^2. In the second case it follows the schema: "If for all s^1 it is the case that $s^1 \leftarrow PX$ and $s^1 \rightarrow s^2$, then $s^2 \leftarrow PX(s^1/s^2)$". Comparison with the schema introduced above is enough to show the difference.

4. CALCULUS

The set of definitions and assertions which contain the terms of abstract objects forms the calculus. In our day one associates with the notion "calculus" the introduction of special symbols, the establishment of sharply specified rules of inference, etc. But this is already quite a refined notion of calculus.

Since the terms of abstract objects do not have empirical correlates the terms themselves are considered as the objects being investigated. And this is justified, for all the definitions and assertions concern the meanings of these terms. In such a view calculi take on the character of formal systems in which the very objects under consideration occur and the rules of reasoning appear as operations with these objects. This is terminologically convenient but it makes the connection with the empirical base even less evident.

5. EMPIRICAL AND EXACT SCIENCES

Abstract objects are formed as means for the investigation of empirical objects (for the obtaining of knowledge about empirical objects). Because of the division of labor in science their formation and investigation is distinguished from that of empirical objects in the form of the development of special sciences which are often called exact or deductive. The interests and needs of the exact sciences have served as the basic stimulus to the development of logic and have to an overwhelming extent determined its content.

There are numerous works dealing with the so-called "problems of the logic and methodology of the deductive (or exact) sciences". Everyone is familiar with the contents of these works: deductive theory (theory of logical inference); theory of proof; axiomatic method and connected questions. We will not deal with these here.

We note that it is impossible to make an absolute division in logic between the problems of the experimental sciences and those of the exact sciences. However, experimental investigation (investigation of empirical objects) has a series of characteristics (as compared with abstract objects) which are fixed in a definite system of concepts and assertions of logic.

6. STATES

D1. Objects designated by the terms

$$\downarrow(s\alpha \leftarrow E) \quad \text{and} \quad \downarrow(s\alpha \leftarrow P)$$

will be called states of the object S. The sentences

$$s\alpha \leftarrow E \quad \text{and} \quad s\alpha \leftarrow P$$

are state-descriptions of S.

D2. Two states are identical only in the following instances:

1) states $\downarrow(s^1\alpha \leftarrow P^1)$ and $\downarrow(s^2\beta \leftarrow P^2)$ are identical if and only if S^1 and S^2, α and β, P^1 and P^2 are pairwise identical;

2) states $\downarrow(s^1\alpha \leftarrow E)$ and $\downarrow(s^2\beta \leftarrow E)$ are identical if and only if S^1 and S^2, α and β are pairwise identical.

If X, Y, X^1, X^2,... are structurally simple sentences, then the symbols

$$x,\ y,\ x^1,\ x^2,\ldots$$

are terms denoting the state-descriptions of the objects mentioned in them.

D3. Two states x and y are disjoint if and only if

$$\sim(X \cdot Y).$$

T1. States $\downarrow(s \leftarrow P)$, $\downarrow(s\urcorner \leftarrow P)$, $\downarrow(s? \leftarrow P)$, $\downarrow(s\urcorner \leftarrow E)$ and $(s? \leftarrow E)$ are pairwise disjoint.

T2. States $\downarrow(s \leftarrow E)$, $\downarrow(s\urcorner \leftarrow E)$ and $\downarrow(s? \leftarrow E)$ are pairwise disjoint.

The existence of states is defined by the assertions:

A1. $(\downarrow(s\alpha \leftarrow P) \leftarrow E) \equiv ((s\alpha \downarrow P) \leftarrow E)$
$(\downarrow(s\alpha \leftarrow E) \leftarrow E) \equiv (s\alpha \leftarrow E)$

$$\begin{aligned} &A2.\ (\downarrow(s\alpha\leftarrow P)\neg\leftarrow E)\equiv(\downarrow(s\beta\leftarrow P)\leftarrow E):(\downarrow(s\gamma\leftarrow P)\leftarrow E)\\ &\quad\ \ (\downarrow(s\alpha\leftarrow E)\neg\leftarrow E)\equiv(\downarrow(s\beta\leftarrow E)\leftarrow E):(\downarrow(s\gamma\leftarrow E)\leftarrow E),\end{aligned}$$

where α, β and γ are distinguished as $\neg$, ? and the absence thereof.

Indeterminacy is excluded for states, i.e.,

$$A3.\ \sim(x\leftarrow E)\equiv(x\neg\leftarrow E).$$

7. SITUATION

D1. A non-empty set of joint states of different (in the case of two and more) objects will be called a situation.

D2. Two situations are different if and only if the sets of their states do not coincide.

D3. Two situations are disjoint if and only if one of them contains at least one state which is disjoint with at least one state of the other.

D4. A situation exists if and only if each of its states exists.

D5. If $X^1,\ldots, X^n$ $(n\geqslant 1)$ are state-descriptions of a given situation, then

$$X^1\cdot\ldots\cdot X^n$$

is a situation-description.

8. THE COLLECTION OF SITUATIONS

D1. A non-empty ordered set of different (in the case of two and more) situations will be called a collection of situations or simply a collection.

In *D1* we find the term "ordered". We will look at this more closely below. Here it will be enough to say that by "ordered" one can have in mind the sequence of situations in time or their distribution in different places. Which particular kind of order is the case is a matter of indifference.

D2. Two collections are different if and only if the ordering of their situations is different or the sets of their situations do not coincide.

D3. A collection exists if and only if each of its situations exists in the proper place in the order.

D4. A collection-description is made up of the situation-descriptions plus additional terms or sentences which fix the order of the situations.

D5. A collection-description is true if and only if all of its situation-descriptions are true in the proper order. Other truth-values for collection-descriptions are introduced according to:

$$([Z^1] \leftarrow v^i) \equiv ([Z^2] \leftarrow v^1),$$

where v^i is a certain truth-value, and Z^2 is a correspondingly selected collection-description, different (if v^i is not v^1) from Z^1.

9. DERIVATIVE SENTENCES

Descriptions of collections (and of situations) form the basis for the construction of those forms of knowledge which are not obtained by inference from other knowledge. All cognitive activities which are not inferential can now be presented as operations with collection-descriptions.

One has to distinguish here: 1) explanation of the origins of certain sentential structures in collection-descriptions; 2) discovery for a given sentence of a collection-description adequate to it. In the first case every structure can theoretically be reduced to a collection-description if idealized conditions are assumed. In the second case one is talking about the practical carrying out of such a reduction (about its expediency). In reality it is often the case that a sentence is reduced to a collection-description only together with other sentences, only in a series of steps, only on different "planes", only to a series of collection-descriptions, etc. In practice this is not always possible and expedient. Only logical analysis of each concrete science can establish how important and possible this is.

Sentences resulting from collection-descriptions can be classified according to the type of collection-descriptions and according to the type of operation used to construct new sentences from them. Different types of collections are known: 1) from one situation and from two or more situations; 2) in each situation only one state or two or more different states; 3) objects and situations are identical and their attributes different; objects different and their attributes identical, etc. Correspondingly, collection-descriptions are also of various types. There are also various types of operations of transition from collection-descriptions to sentences. In some cases the operations are such that one can according to the terms or logical signs in the sentences establish the collection-descriptions from which they are obtained; in other cases one can establish only the logical type of the collection (so that new observations are needed in order to obtain the collection-description). As a result of the combination of the attributes indicated one obtains sentences which cannot be classified solely according to one attribute.

Thus one has to introduce other principles, relative to which classification becomes a secondary affair.

Sentences which result from collection-descriptions contain terms or logical signs which make it possible to reconstruct the descriptions themselves or to establish their logical type. If this involves transformation of terms into new terms, then the sentences must include signs which refer to the corresponding operations.

The construction of sentences from collection-descriptions is substitution for the latter. Such substitution meets

$$([X] \leftarrow v^1) \rightarrow ([Y] \leftarrow v^1),$$

where Y is a sentence replacing the collection-description X. Verification of sentence Y is carried out as follows: depending on the character of Y, either one establishes X and carries out its verification or one explains which type of collection-description is to be obtained. The second case requires supplementary investigations which produce some collection-description X^*. On the basis of existing agreements one gets Y^* from X^* and from the comparison of Y and Y^* one establishes the truth-value of the first.

For example, let $a = f(b)$; in view of the type of f we can decide that a collection-description of the type $(((a = \dots), (b = \dots)), \dots, ((a = \dots), (b = \dots)))$ has taken place; for the confirmation of the sentence it is necessary to fill the empty places, i.e., to carry out the corresponding observations and measurements, and then to use rules making it possible to replace this set of sentences with a sentence of the type $a = f(b)$, i.e., to establish a mathematical type of dependence; if we get $a = f(b)$ or $a = f^*(b)$ such that for us there is practically no difference between f and f^*, then $a = f^*(b)$ is taken as true; otherwise, it is false. When one says that $a = f(b)$ is confirmed in practice, this is what is factually done: through a given value b one finds a on the basis of $a = f(b)$ and sees that the result coincides with what was observed; and so on; but each time one fixes the observed values of a and b, i.e., implicitly defines a collection-description.

From collection-descriptions one obtains sentences which contain terms or logical signs lacking in the former. We will repeatedly have to do with such cases in what follows.

10. VARIATION

D1. The states

$$\downarrow(s \leftarrow E), \downarrow(s \urcorner \leftarrow E), \downarrow(s \leftarrow P), \downarrow(s \urcorner \leftarrow P)$$

are static states and

$$\downarrow(s? \leftarrow E), \downarrow(s? \leftarrow P)$$

are transitional states.

Static states will be represented by the symbols

$$x, y, x^1, \ldots, \urcorner x, \urcorner y, \urcorner x^1, \ldots$$

and transitional states by

$$?x, ?y, ?x^1, \ldots.$$

These symbols are related as follows: if one (no matter which) of x and $\urcorner x$ is $\downarrow(s \leftarrow E)$, then the other is $\downarrow(s \urcorner \leftarrow E)$; $?x$ here is $\downarrow(s? \leftarrow E)$; similarly for $\downarrow(s\alpha \leftarrow P)$.

A1. $\urcorner \urcorner x \equiv x.$

Empirically given is the fact that the investigator cannot observe the states x, $\urcorner x$ and $?x$ simultaneously; he can observe them only separately.

D2. That an elementary variation affecting s happens means to observe a situation in which state x takes place (s is in X) and then to observe a situation in which $\urcorner x$ takes place. We indicate this with the symbols

$$x \Rightarrow \urcorner x.$$

Partial negation of it means: to observe a situation with $\downarrow x$ and then a situation but again with $\downarrow x$. We indicate this with

$$x \Rightarrow x.$$

A2. $(x \urcorner \Rightarrow \urcorner x) \equiv (x \Rightarrow x)$

A3. $\sim (x \Rightarrow \urcorner x) \equiv (x \urcorner \Rightarrow \urcorner x).$

D3. Elementary variation is a sequence of situations x, $?x$ and $\neg x$. Symbolically,

$$(x, ?x, \neg x).$$

D4. Two elementary variations $(x, ?x, \neg x)$ and $(y, ?y, \neg y)$ are different if and only if the states x and y are different.

D5. A variation is a non-empty ordered set of elementary variations. All that pertains to elementary variations is extended to variations in general by generalizing the symbolization.

Investigation of variation proceeds as follows:

1) one fixes that which varies;

2) one fixes the transitional state from one (such) static state to another;

3) an analysis of the variation is performed, the result of which is that the variations composing it are explained;

4) in its turn the transitional state is studied as a variation.

D6. A variation $(x, ?x, \neg x)$ is considered discrete if attention is not paid to $?x$. The limit case is when one assumes that there is no $?x$ (and we then have to do with an abstract object).

D7. A variation $(x, ?x, \neg x)$ is considered not discrete if attention is paid to $?x$ and the latter in its turn is considered as a set of not discrete variations. Theoretically such a process of considering a transitional state as a set of variations is endless. In practice, however, a limit is always set, so that certain variations are always considered as elementary and their states are not subject to further analysis.

11. VARIATION OF ATTRIBUTES

D1. Attributes $P^1, \ldots, P^n$ $(n \geqslant 2)$ are variants of an attribute P if and only if

$$P \rightharpoonup P^1, \ldots, P \rightharpoonup P^n,$$

and for any S

$$\sim((s \leftarrow P^i) \cdot (s \leftarrow P^k)),$$

where P^i and P^k are any pair from $P^1, \ldots, P^n$. For example, the attribute "moves with speed a^1" and "moves with speed a^2" are variants of "moves", if $a^1 \neq a^2$.

D2. The set of attributes satisfying *D1* forms the range of variation of the given attribute. One and the same attribute can have two and more different ranges of variation. Thus, the variants of the attribute "moves" are the attributes "moves left" and "moves right" relative to a different range of variation than the attributes "moves with the speed a^1" and "moves with speed a^2".

T1. If P^2 is a variant of P^1 and $s \leftarrow P^2$ (or $s \urcorner \leftarrow P^1$; or $s? \leftarrow P^1$), then (according to *D1* and *T4*VIII4) $s \leftarrow P^1$ (correspondingly $s \urcorner \leftarrow P^2$; $s? \leftarrow P^2$).

T2. According to *D4*IX3 the sentence $s \leftarrow P$ is true relative to any $S \downarrow \mathrm{P}^i$, where P^i is a variant of P.

The opposite of *T2* is also possible: two and more different sentences $s \leftarrow P^k$ can be considered true relative to one and the same $S \downarrow \mathrm{P}^i$.

The terms of the variants of P are formed as follows:

1) there are objects $\delta^1, \delta^2, \ldots$, such that their joining to P provides

$$P\delta^i,$$

designating variants of P;

2) if by agreement

$$s^* \leftarrow P^2 \equiv s^* \leftarrow P^1 \delta^i,$$

then the term P^2 is a term of a variant of P.

T3. If P^1 is a variant of P^2 then for any S

$$(s \leftarrow P^1) \rightarrow (s \leftarrow P^2)$$
$$(s \urcorner \leftarrow P^2) \rightarrow (s \urcorner \leftarrow P^1).$$

The formation of the term $P\,\delta^i$ is not the formation of a complex term according to the rules of logic. It is (in our sense) a simple term. The objects δ^i are considered autonomous signs only to the extent that it is implicitly assumed that they are parts of $P\,\delta^i$.

12. MAGNITUDE

A special case of terms of the type $P\,\delta^i$ includes terms in which δ^i is an attribute of magnitude. Attributes of magnitude include such archaic expressions as "many", "strongly", "slowly", etc. In science signs of magnitude are numbers with names of units of measurement and signs of the methods of obtaining numbers. In other words, they have a complex structure. The corresponding terms consist of three parts and have the form

$$P\alpha\beta,$$

where P is the name of an attribute, α is a number and β the name of a magnitude. The last, in its turn, can be complex.

Magnitudes are simple (e.g., "5 kg") and derivative (e.g., "10 kg/m"). A measurement can be direct or indirect. In a direct measurement $s \leftarrow P\alpha\beta$ is obtained immediately. In the indirect measurement one obtains $s^2 \leftarrow P^2\alpha\beta$ and on the basis that

$$(s^1 \leftarrow P^1) \rightarrow (s^2 \leftarrow P^2\alpha\beta),$$

$s^1 \leftarrow P^1\alpha\beta$ is accepted.

Measurements can be divided into two groups according to the following attribute. In some cases one has completely determined units of measurement and there are standard measuring procedures. In other cases there are no determined units of measurements (some abstract units are simply assumed) and there are no standard procedures of measurement (one simply assumes that there are such procedures). Some number of units is assigned to attributes. The considerations which determine this depend on the concrete particularities of the problem and of the conditions for its solution. Only experience can provide objective criteria (the assigning of this magnitude to this attribute used to give this result).

Normally one understands measurement in the first sense. The second type of measurement (as distinguished from the first, from measurement in the strict sense) could be called the "weighting of attributes" (in the sense of finding a numerical "weight" for attributes).

There are cases where all $\delta^1,\ldots,\delta^n$ are different and all $s \leftarrow P\delta^1,\ldots,$ $s \leftarrow P\delta^n$ are considered true. Differences between such sentences, if they are of importance, are fixed by signs of degree of accuracy and approximation.

13. RANGE OF TRUTH

D1. The set of all possible $s \leftarrow P^i$ $(i \geqslant 1)$ which are considered true relative to one and the same object $S \downarrow P^i$ forms the range of truth of the sentences about this object.

In this case the attribute P appears as a variant of two or more different attributes $P^1, P^2, \ldots$. For example, the sentences "particle a moves with speed b^1" and "particle a moves with speed b^2" can both be considered true relative to the moving particle a, though $b^1 \neq b^2$. More often than not a range of truth is determined by means of a relation of magnitudes (e.g., magnitudes a and b are assigned so that if $a \leqslant \delta^i \leqslant b$, then $s \leftarrow P\delta^i$ is considered true).

The range of truth is established in every domain of science in function of its possibilities and needs. There are no logical criteria for this unless we consider the banal desiderata that the range be limited and that it not include all possible sentences. It is only because logical pedantry gives way to practical expediency that in many cases it is possible to use sentences and from the assumption of what is not and cannot be to obtain sentences which are true in the range adopted.

CHAPTER TEN

SENTENCES WITH QUANTIFIERS

1. QUANTIFIERS

Normally the signs "all" and "some" as used in sentences together with their terms are what one has in mind as quantifiers. Here we will also include the signs "zero", "one", "two",..., "infinite number", etc. and the signs der ivedfrom them ("majority", "minority", "a half", "a third", etc.).

Quantifiers are not terms of sentences or even parts of terms, although the words "one", "two", etc. can be terms or parts of terms. For example, in the sentence "Three types of contemporary plane fly faster than three thousand kilometers per hour" the word "three" is a quantifier in the first case but a part of the predicate in the second. Thus, quantifiers are not simply words like "one", "two", etc., but certain functions of signs of a certain type.

We admit the designations:

1) ꓘ is any quantifier;
2) ꓘt indicates ꓘ of objects t;
3) ¬ꓘt indicates no ꓘ of objects t;
4) ?ꓘt indicates that it is impossible to establish ꓘt or ¬ꓘt.

Logic deals in detail only with the properties of the quantifiers "all" ("every") and "some" ("at least one"). We will designate them as $\forall$ and $\exists$.

2. THE STRUCTURE OF QUANTIFIED SENTENCES

In logic the way of forming sentences is of great importance since it involves their logical standardization, i.e., the matter of their structure. This is very clear in the case of quantified sentences.

In natural languages (both ordinary and scientific) the quantifiers usually stand directly before the terms of the sentence, as in "All *a* are larger than some *b*". But it is accepted practice in logic to extract the quantifiers from the sentences and to write them on the same line with the sentences, indicating the terms to which they apply. Thus, if X is a sentence, t a term, and α indicates that before $>\!\!|$ there is or is not one of the signs $\neg$ and ?, then the sentence containing $\alpha >\!\!|$ is written in the form $(\alpha >\!\!| t)X$. The convenience of this is evident: one can consider sentences with any structure (i.e., one can abstract from the structure of X) and formulate quite general rules for sentences with quantifiers.

But this is not all. The writing of the quantifiers directly before the terms has one great disadvantage: in some cases sentences with two or more quantifiers allow of more than one interpretation. Thus, the sentence "All *a* are larger than some *b*" can be read as "For all *a* there is some *b* such that *a* is larger than *b*" and as "Some *b* are such that all *a* are larger than them". These interpretations are not identical in meaning (the second sentence does not follow from the first; the first could be true while the second was false). This indicates that in constructing sentences with two or more quantifiers one must pay attention not only to the type of quantifier and the terms before which they are placed but also to the order in which they are written. And the order of the quantifiers might not agree with the order of the terms in the sentence. Differences in the order of writing the quantifiers can mean differences in the meaning of the sentences obtained in different acts of investigation. The way of writing quantified sentences in logic takes this into account. This is recommended to science as a way of obviating ambiguity.

We noted above that one and the same sign can play the role both of quantifier and of part of a term. The logical standardization of sentences makes it possible to distinguish the two occurrences of a sign. Let us take, for example, the sentence "Two atoms of hydrogen and one atom of oxygen form a molecule of water". We put it in standard form: "Two atoms of hydrogen and one atom of oxygen are such that the atom of hydrogen and atom of oxygen form a molecule of water". Clearly we get a false sentence since we took the words "one" and "two" as quantifiers which they are not.

Finally, the mode of presentation of quantified sentences in logic makes their logical properties much clearer and this is important for science. As we have seen, even on the relatively trivial level of the matter of writing in a suitable way logic plays a role in perfecting the language of science.

We will use the symbolism accepted in logic with a few corrections.

The inclusion of a quantifier in sentence X will follow the schema:

1) by the transformation rules we obtain $t \leftarrow PX$ from X;

2) the results of the investigation determine the attribution of $\alpha{>\!\!|}$ to the term t; we obtain $\alpha{>\!\!|}t \leftarrow PX$ which reads as "$\alpha{>\!\!|}$ elements of the class t are such that X" or simply "$\alpha{>\!\!|}t$ are such that X" (also reading "X is true relative to $\alpha{>\!\!|}$ objects of Kt" or simply "X is true in relation to $\alpha{>\!\!|}t$")

Taking the formula

$$\alpha{>\!\!|}t \leftarrow PX$$

as primitively explicit we adopt the following stipulations, playing a purely formal role:

A1. $(\alpha{>\!\!|}t)(t \leftarrow PX) \equiv \alpha{>\!\!|}t \leftarrow PX$

A2. $(\alpha{>\!\!|}t)X \equiv (\alpha{>\!\!|}t)(t \leftarrow PX)$.

Let ${>\!\!|}^1, {>\!\!|}^2, \ldots$ each be singly any quantifier; if they are found together (in one assertion), then the difference of superscripts means only that the quantifiers can be pairwise different and identical in any possible combinations. Further, let each of the signs $\alpha^1, \alpha^2, \ldots$ before a quantifier singly indicate that before this quantifier there is or is not any of the signs $\urcorner$ and ?; if $\alpha^1, \alpha^2, \ldots$ are found together (in one assertion) then the difference of superscripts indicates only that the signs before the quantifiers can be different and identical (and lacking) in any combination. Sentences with several quantifiers will be written in the form

$$(\alpha^1{>\!\!|}^1t^1)((\alpha^2{>\!\!|}^2t^2)X)$$
$$(\alpha^1{>\!\!|}^1t^1)((\alpha^2{>\!\!|}^2t^2)\ldots((\alpha^n{>\!\!|}^nt^n)X)\ldots).$$

Following the stipulations, they will be read as

$$\alpha^1{>\!\!|}^1t^1 \leftarrow P(\alpha^2{>\!\!|}^2t^2 \leftarrow PX)$$
$$\alpha^1{>\!\!|}^1t^1 \leftarrow P(\alpha^2{>\!\!|}^2t^2 \leftarrow P(\ldots \leftarrow P(\alpha^n{>\!\!|}^nt^n) \leftarrow PX)\ldots).$$

For simplicity we adopt:

A3. $(\alpha^1{>\!\!|}^1t^1)(\alpha^2{>\!\!|}^2t^2)X \equiv (\alpha^1{>\!\!|}^1t^1)((\alpha^2{>\!\!|}^2t^2)X)$.

T1. $(\alpha^1{>\!\!|}^1t^1)(\alpha^2{>\!\!|}^2t^2)\ldots(\alpha^n{>\!\!|}^nt^n)X \equiv (\alpha^1{>\!\!|}^1t^1)((\alpha^2{>\!\!|}^2t^2)\ldots$
$\ldots((\alpha^n{>\!\!|}^nt^n)X)\ldots)$

follows from *A3*.

If Y is $(\alpha^1{>\!\!|}^1t^1)X$ or $(\alpha^1{>\!\!|}^1t^1)\ldots(\alpha^n{>\!\!|}^nt^n)X$, where $n \geqslant 2$ and X does not contain quantifiers, we will call X an unquantified basic sentence of Y and $(\alpha^1 >\!\!|^1 t^1) \ldots (\alpha^n >\!\!|^n t^n)$ a quantifier group (or prefix) of Y. If there are no quantifiers in Y then we will say that it has an empty quantifier group.

D1. To the definition of a sentence we can now add a point having to do with quantifiers: if X is a sentence, then $(\alpha >\!\!| t)X$ is a sentence;

We also adopt the following complements to the definition of occurrence:

D2. 1) $\alpha >\!\!| t$ occurs in $(\alpha >\!\!| t)X$; t occurs in $\alpha >\!\!| t$;

2) X occurs in $(\alpha >\!\!| t)X$;

3) if $\alpha >\!\!| t$ occurs in X and X occurs in Y then $\alpha >\!\!| t$ occurs in Y.

D3. If t occurs in X, then t is bound (occurs as bound) in $(\alpha >\!\!| t)X$; if t occurs as bound in X and X occurs in Y then Y has a bound occurrence of t.

D4. If t occurs in X and $\alpha >\!\!| t$ does not, then t freely occurs in X.

3. INDETERMINACY

Let it be impossible to prove or disprove $(\forall s)X$ and to sort out all S (because, for example, the number of s is infinite). Let all S examined be such that X. No matter how great the number of cases examined, there is no logical basis for accepting $(\forall s)X$ or $(\neg \forall s)X$. This situation can be indicated with the help of the sign of indeterminacy, i.e., $(? \forall s)X$. And if in science in such cases $(\forall s)X$ is nevertheless accepted this is not for logical reasons but for other considerations (e.g., enough cases are examined; the sentence is taken as a hypothesis, etc.). Thus an examination of the possibility of cases with a sign of indeterminacy before the quantifier is fully justified in a logical investigation of the language of science. The indeterminacy of the quantifier is different from that which might be found in the base. What is more, these indeterminacies are completely independent.

4. QUANTIFICATION OF TERMS

The construction of sentences, the base of which contains the term t with $\alpha >\!\!| t$ occurring in the quantifier group, is the quantification of the term t.

In science the quantification of subjects is usual. Logic as a science began with the investigation of this (Aristotelian syllogistics). As to the quantification of predicates, this seems for a number of reasons obscure and unnatural at first glance. It is no accident that it has been taken up by logic only in modern times.

The reasons are as follows. More often than not attributes are not distinguished by type and are not quantified (are not distributed into parts) so that the conditions for application of quantifiers were simply not there: quantifiers were used only when it was a matter of classes. If the conditions are given, and quantification of predicates is carried out then the attributes fixed by them become objects alongside the objects fixed by subjects. One then has to do with classes of objects which can be attributes of other objects. For example, in the sentence "*a* has a certain degree of freedom" the degrees of freedom are considered objects differing in type. If P is the predicate of sentence X then in quantification it becomes one of the terms of the subject, and the predicate of the newly obtained sentence becomes an expression like "the second (or the first, depending on the place of P) is an attribute of the first (or the second)". Thus, in the quantification of the predicate in $s \leftarrow P$ we obtain a sentence with the base $(s, P) \leftarrow Q$, where Q is the expression "the second is an attribute of the first".

Finally, if the attributes differ in type then the quantification is implicit, since one accepts

$$(s \leftarrow P) \rightarrow (\exists P)(s \leftarrow P);$$
$$(s \neg\leftarrow P) \rightarrow (\forall P)(s \neg\leftarrow P);$$
$$(s? \leftarrow P) \rightarrow (\forall P)(s? \leftarrow P).$$

Since everything that is true for the quantification of subjects is true for the quantification of predicates (but not vice versa!) in what follows we will speak simply of the quantification of any terms.

5. EXTRINSIC NEGATION

D1. $\sim(\alpha^1 \text{ꓘ} t)\, X \equiv (\alpha^2 \text{ꓘ} t)\, X : (\alpha^3 \text{ꓘ} t)\, X,$

where α^1, α^2 and α^3 are different.

The classical case

$$\sim(\text{ꓘ} t)\, X \equiv (\neg \text{ꓘ} t)\, X$$
$$\sim(\neg \text{ꓘ} t)\, X \equiv (\text{ꓘ} t)\, X$$

occurs only when the possibility of $?\text{Ʞ}$ is excluded. Thus the assertions

$$\sim(\neg\,\text{Ʞ}t)\,X \vdash (\text{Ʞ}t)\,X$$

including

$$\sim(\neg\,\forall t)\,X \vdash (\forall t)\,X$$
$$\sim(\neg\,\exists t)\,X \vdash (\exists t)\,X$$

are invalid in the general (non-classical) case. They are true only for the classical case. If they are accepted, this is equivalent to exclusion of indeterminacy of quantifiers.

6. DEFINITIONS OF QUANTIFIERS

Let

$$t1, t2, \ldots tA$$

be all possible individual terms of the value-range of t and the number A is the power of Kt. The number A can be infinite but this should not be confusing: we are talking here not about obtaining true sentences with quantifiers but only about defining the sense of the latter. Each ti singly is some (no matter which) individual term of the value-range of t and a difference of jointly taken ti's means only that these terms are different in value.

The symbol

$$Xi$$

will designate a sentence formed from X by putting ti for t wherever t occurs in X. Let t not occur bound in X and be the only free term in X.

The quantifiers ∀ and ∃ are usually defined as:

$$(\forall t)\,X \equiv X1\cdot\ldots\cdot X\mathrm{A}$$
$$(\exists t)\,X \equiv X1 \vee \ldots \vee X\mathrm{A}.$$

These definitions are correct since A is finite and there is no possibility of indeterminacy. But if A is infinite (or just sufficiently large to be practically infinite, etc.) then the definition will deal not with the Xi themselves but merely with the putative possibility of their construction. Then one would have to take into account cases where it is not possible to construct such sentences for all individuals of the value-range of the quantified term, i.e., where there is indeterminacy of quantifiers. And if the latter is recognized as possible, then the definitions introduced are contradictory. In fact the assertions

$$
\begin{array}{l}
(?\forall t)X \vdash \sim(\forall t)X\cdot\sim(\neg\,\forall t)X \\
(\neg\,\forall t)X \dashv\vdash (\exists t)\sim X \\
\sim(\sim X1 \vee \ldots \vee \sim X\mathrm{A}) \vdash X1\cdot\ldots\cdot X\mathrm{A}
\end{array}
$$

are true. From them we obtain

$$(?\forall t)X \vdash \sim(X1\cdot\ldots\cdot X\mathrm{A})\cdot(X1\cdot\ldots\cdot X\mathrm{A}).$$

Similarly we obtain

$$(?\exists t)X \vdash \sim(X1 \vee \ldots \vee X\mathrm{A})\cdot(X1 \vee \ldots \vee X\mathrm{A}).$$

To avoid this it is necessary either to abandon the indeterminacy of quantifiers or to make some corrections in their definitions.

We adopt the designation:

$\{Y\}$ for : it is possible to construct a true Y;

We adopt the following assertions to define the sense of quantifiers:

$$
\begin{array}{ll}
D1. & (\exists t)X \equiv \{X1\} \vee \ldots \vee \{X\mathrm{A}\} \\
 & (\neg\exists t)X \equiv \{\sim X1\}\cdot\ldots\cdot\{\sim X\mathrm{A}\} \\
 & (?\exists t)X \equiv \sim(\exists t)X\cdot\sim(\neg\exists t)X \\
D2. & (\forall t)X \equiv \{X1\}\cdot\ldots\cdot\{X\mathrm{A}\} \\
 & (\neg\forall t)X \equiv \{\sim X1\} \vee \ldots \vee \{\sim X\mathrm{A}\} \\
 & (?\forall t)X \equiv \sim(\forall t)X \sim (\neg\forall t)X
\end{array}
$$

Other variants are possible, such as:

Variant I: we assume *D2* and instead of *D1* we assume

$$
\begin{array}{ll}
D'1. & (\exists t)X \equiv (\neg\,\forall t)\sim X \\
 & (\neg\,\exists t)X \equiv (\forall t)\sim X \\
 & (?\exists t)X \equiv (?\forall t)\sim X.
\end{array}
$$

Variant II: we assume *D1* and instead of *D2* we assume

$$
\begin{array}{ll}
D'2. & (\forall t)X \equiv (\neg\,\exists t)\sim X \\
 & (\neg\,\forall t)X \equiv (\exists t)\sim X \\
 & (?\forall t)X \equiv (?\exists t)\sim X.
\end{array}
$$

The relations indicated in *D'1* and *D'2* are obtained from *D1* and *D2* by substituting Xi for $\sim\sim Xi$.

The classical case will have the following form:

D_1^c. $(\exists t)\,X \equiv \{X1\} \vee \ldots \vee \{X\mathrm{A}\}$

$\sim(\exists t)\,X \equiv \{\sim X1\}\cdot\ldots\cdot\{\sim X\mathrm{A}\}$

D_2^c. $(\forall t)\,X \equiv \{X1\}\cdot\ldots\cdot\{X\mathrm{A}\}$

$\sim(\forall t)\,X \equiv \{\sim X1\} \vee \ldots \vee \{\sim X\mathrm{A}\}$

And since the { } are superfluous, we obtain the definitions indicated in the beginning of the section.

7. OTHER QUANTIFIERS

Let N be any natural number. The following definitions of $\alpha\mathrm{N}t$ are possible

D1. $(\mathrm{N}t)\,X \equiv \{X1\}\cdot\ldots\cdot\{X\mathrm{N}\}$

$(\urcorner\,\mathrm{N}t)\,X \equiv \{\sim X1\}\cdot\ldots\cdot\{\sim X(\mathrm{A}-\mathrm{N}+1)\}$

$(?\,\mathrm{N}t)\,X \equiv \sim(\mathrm{N}t)\,X\cdot\sim(\urcorner\,\mathrm{N}t)\,X$

(they are not the only ones possible).

T1. $((\mathrm{N}+1)\,t)\,X \vdash (\mathrm{N}t)\,X$

T2. $(\urcorner\,\mathrm{N}t)\,X \vdash (\urcorner\,(\mathrm{N}+1)\,t)\,X$

T3. $(?\,\mathrm{N}t)\,X \vdash (?(\mathrm{N}+1)\,t)\,X$.

In *D1* the quantifier N is defined in the sense "At least N". A restrictive definition ("N and only N" which we designate as N^0) has the form

D2. $(\mathrm{N}^0t)\,X \equiv (\mathrm{N}t)\,X\cdot(\urcorner\,(\mathrm{N}+1)\,t)\,X$

$(\urcorner\,\mathrm{N}^0t)\,X \equiv (\mathrm{N}t)\,X\cdot((\mathrm{N}+1)\,t)\,X$

$(?\,\mathrm{N}^0t)\,X \equiv (\mathrm{N}t)\,X\cdot(?(\mathrm{N}+1)\,t)\,X$.

Obviously,

T4. $(\alpha\mathrm{N}^0t)\,X \vdash (\mathrm{N}t)\,X$.

It is only in the case $(\mathrm{N}^0t)X$ that the quantifier N coincides with the number of objects t such that X.

The quantifier "zero" is defined by

D3. $(0t)\,X \equiv (\urcorner\,1t)\,X$.

$(\urcorner\,0t)\,X \equiv (1t)\,X$

$(?\,0t)\,X \equiv (?\,1t)\,X$.

If $(N^1 t)X$ and $(N^2 t)X$, then such quantifiers as "majority", "minority", "third", "almost all", etc., are introduced, depending on the relations between N^1, N^2 and A.

The quantifier "some and only some" (we designate it as $\exists^0$) is defined by the assertions:

D4. $(\exists^0 t)\,X \equiv (\exists t)\,X \cdot (\neg\,\forall t)\,X$
$(\neg\,\exists^0 t)\,X \equiv (\forall t)\,X$
$(?\exists^0 t)\,X \equiv (\exists t)\,X \cdot (?\forall t)\,X.$

Obviously,

T5. $(\alpha\exists^0 t)\,X \vdash (\exists t)\,X.$

8. A NUMBER OF QUANTIFIERS

The assertions admitted previously

$$X \equiv (t \leftarrow PX)$$
$$\sim (t \leftarrow PX) \equiv (t \leftarrow P \sim X)$$
$$(\alpha^1\text{ꓘ}^1 t^1)\,(\alpha^2\text{ꓘ}^2 t^2)\,X \equiv (\alpha^1\text{ꓘ}^1 t^1)\,((\alpha^2\text{ꓘ}^2 t^2)\,X)$$

and the definitions of quantifiers

T1. $(\text{ꓘ}t)\,(\text{ꓘ}t)\,X \leftrightarrow (\text{ꓘ}t)\,X$
T2. $(\exists t^1)\,(\exists t^2)\,X \leftrightarrow (\exists t^2)\,(\exists t^1)\,X$
T3. $(\forall t^1)\,(\forall t^2)\,X \leftrightarrow (\forall t^2)\,(\forall t^1)\,X$
T4. $(\exists t^1)\,(\forall t^2)\,X \rightarrow (\forall t^2)\,(\exists t^1)\,X$
T5. $(\exists t^1)\,(\exists t^2)\,X \rightarrow (\exists\,(t^1, t^2))\,X$
T6. $(\forall t^1)\,(\forall t^2)\,X \rightarrow (\forall\,(t^1, t^2))\,X$
T7. $(\forall t^1)\,(\exists t^2)\,X \rightarrow (\exists\,(t^1, t^2))\,X$
T8. $(\exists t^1)\,(\forall t^2)\,X \rightarrow (\exists\,(t^1, t^2))\,X$
T9. $(\exists t^1)\,(?\exists t^2)\,X \rightarrow (?\exists\,(t^1, t^2))\,X$
T10. $(?\exists t^1)\,(\exists t^2)\,X \rightarrow (?\exists\,(t^1, t^2))\,X$
T11. $(\forall t^1)\,(?\exists t^2)\,X \rightarrow (?\exists\,(t^1, t^2))\,X$
T12. $(?\exists t^1)\,(\forall t^2)\,X \rightarrow (?\exists\,(t^1, t^2))\,X$
T13. $(\forall t^1)\,(?\forall t^2)\,X \rightarrow (?\forall\,(t^1, t^2))\,X$
T14. $(?\forall t^1)\,(\forall t^2)\,X \rightarrow (?\forall\,(t^1, t^2))\,X$
T15. $(?\forall t^1)\,(?\forall t^2)\,X \rightarrow (?\forall\,(t^1, t^2))\,X$
T16. $(\exists t^1)\,(\neg\,\exists t^2)\,X \rightarrow (\exists\,(t^1, t^2)) \sim X.$

are sufficient for obtaining assertions relative to cases of two and more quantifiers.

The assertion

$$(\forall t^1)(\exists t^2)X \rightarrow (\exists t^2)(\forall t^1)X$$

is true only if the class t^1 or t^2 contain one member. In general it is not true. It can happen that

$$\begin{aligned}(\forall t^1)(\exists t^2)X &\equiv ((X(t^2/t^21) \vee X(t^2/t^22))(t^1/t^11)(X(t^2/t^21) \vee \\ &\qquad \vee X(t^2/t^22))(t^1/t^12))\\ (\exists t^2)(\forall t^1)X &\equiv ((X(t^1/t^11)\,X(t^1/t^12))(t^2/t^21) \vee \\ &\qquad \vee (X(t^1/t^11)\,X(t^1/t^12))(t^2/t^22)).\end{aligned}$$

It is impossible to obtain $(\exists t^2)(\forall t^1)X$ from $(\forall t^1)(\exists t^2)X$.

In classical logic the rules

$$\begin{aligned}&\sim((\forall t^1)(\forall t^2)X) \rightarrow (\exists t^1)(\exists t^2) \sim X\\ &\sim((\exists t^1)(\forall t^2)X) \rightarrow (\forall t^1)(\exists t^2) \sim X,\end{aligned}$$

etc., hold. According to our approach, such assertions hold only for the classical cases (without indeterminacy). In general, however, they are not true. Thus,

$$\begin{aligned}&\sim((\forall t^1)(\exists t^2)X) \rightarrow (\exists t^1)((\forall t^2) \sim X:(?\forall t^2) \sim X):\\ &\qquad :(?\exists t^1)((\forall t^2) \sim X:(?\forall t^2) \sim X)\\ &\sim((\forall t^1)(\forall t^2)X) \rightarrow (\exists t^1)((\exists t^2) \sim X:(?\exists t^2) \sim X):\\ &\qquad :(?\exists t^1)((\exists t^2) \sim X:(?\exists t^2) \sim X),\end{aligned}$$

etc. Even for intrinsic negation such assertions are in general not true. Thus,

$$\begin{aligned}&(\neg\forall t^1)(\exists t^2)X \rightarrow (\exists t^1)((\forall t^2) \sim X:(?\forall t^2) \sim X)\\ &(\neg\forall t^1)(\forall t^2)X \rightarrow (\exists t^1)((\exists t^2) \sim X:(?\exists t^2) \sim X)\end{aligned}$$

9. TRUTH-VALUES

D1. $[(\forall t)X] \leftarrow v^1$ if and only if it is possible to construct each of Xi and each of them is true; $[(\neg\forall t)X] \leftarrow v^1$ if and only if it is possible to construct at least one not-true Xi; $[(?\forall t)X] \leftarrow v^1$ if and only if it is impossible to construct at least one of Xi and all Xi that can be constructed are true.

D2. $[(\mathrm{N}t)X] \leftarrow v^1$ if and only if it is possible to construct at least N true Xi; $[(\neg \mathrm{N}t)X] \leftarrow v^1$ if and only if it is possible to construct A−N+1 not-true Xi; $[(?\mathrm{N}t)X] \leftarrow v^1$ if and only if it is impossible to construct N true Xi.

D3. $[(\exists t)\,X] \leftarrow v^1 \equiv [(\neg\,\forall t) \sim X] \leftarrow v^1$
$[(\neg\,\exists t)\,X] \leftarrow v^1 \equiv [(\forall t) \sim X] \leftarrow v^1$
$[(?\,\exists t)\,X] \leftarrow v^1 \equiv [(?\,\forall t) \sim X] \leftarrow v^1$.

For the other quantifiers v^1 is defined, depending on their definitions. The other truth-values are defined for any $\rtimes$ as:

D4. $[(\rtimes t)\,X] \leftarrow v^4 \equiv [(\neg \rtimes t)\,X] \leftarrow v^1$
$[(\neg \rtimes t)\,X] \leftarrow v^4 \equiv [(\rtimes t)\,X] \leftarrow v^1$
$[(? \rtimes t)\,X] \leftarrow v^4 \equiv [(\rtimes t)\,X] \leftarrow v^1 : [(\neg \rtimes t)\,X] \leftarrow v^1$
D5. $[(\rtimes t)\,X] \leftarrow v^2 \equiv [(? \rtimes t)\,X] \leftarrow v^1$
$[(\neg \rtimes t)\,X] \leftarrow v^2 \equiv [(? \rtimes t)\,X] \leftarrow v^1$

If E does not occur in X, then:

D6. $[(\alpha \rtimes t)\,X] \leftarrow v^3 \equiv [\sim (t \leftarrow E)] \leftarrow v^1$.

In order to decide if sentences are tautologies and for comparing them as to truth conditions, it is enough to accept the following relations:

A1. $(\alpha \rtimes t)X \approx Y$, when Y is the expression to the right of $\equiv$ in the definitions of Section 6.

A2. If one of $\{X\}$ and $\{\sim X\}$ takes the value v^1, then the other takes the value nv^1; but both of them can take the value nv^1, i.e., if one of them takes the value nv^1, then the other can take v^1 or nv^1 (both possibilities have to be considered; in other words, if one of them takes the value nv^1, then the value of the other does not depend on the value of the first). The value-relation formulated here (in *A2*) does not influence the fact that $X \approx\, \sim\, \sim X$ in all cases.

T1. It follows from *A2* that both $\sim\{X\}$ and $\sim\{\sim X\}$ can have the value v^1.

10. QUANTIFIERS AND EXISTENCE

The signs E and $\exists$ are different already because $\alpha\exists t$ is not a sentence, which $t\beta \leftarrow E$ is. What is more, the sentences $(\alpha \rtimes t)$ $(t\beta \leftarrow E)$ are possible. Let us imagine the following situation: one investigator on the basis of certain considerations asserts that individuals $t1, t2, \ldots, tn$ have certain attributes and another investigator asserts that some of these individuals do not exist at all, i.e., he constructs the sentence $(\exists t)$ $(t \neg \leftarrow E)$.

If E does not occur in X then, by virtue of the definitions of truth-values, quantifiers and the signs $\cdot$, :, $\sim$, etc., the assertions

T1. $([(\exists t)X] \leftarrow v^1) \rightarrow ((t \downarrow PX) \leftarrow E).$
T2. $([(\exists t)X] \leftarrow v^1) \rightarrow (t \leftarrow E).$
T3. $([(\forall t)X] \leftarrow v^1) \rightarrow (t \leftarrow E)$

will be true. This creates the illusion that in order to operate with quantifiers it is necessary that Kt not be existentially empty (or as one sometimes says: that the object domain not be empty). In science, however, one sometimes has to do with not-true (false, indeterminate, unprovable) sentences with quantifiers, for which Kt is existentially empty.

11. RULES OF LOGICAL ENTAILMENT

A system of rules of logical entailment for quantified sentences is formed directly from the definitions (by putting $\dashv\vdash$ for $\equiv$) and indirectly as follows: from the analysis of previously accepted definitions and assertions one can obtain a series of assertions on relations of sentences from the point of view of truth-value and units of meaning and, on this basis, one can formulate a system of assertions $X \vdash Y$ and $\vdash X$ (or for the weak entailment). Below we will indicate the direction of construction of an intuitive base for a theory of logical entailment in such a case.

In the construction of a theory of logical entailment for sentences with quantifiers we adduce truth-value stipulations on the relations between premisses and conclusion. Stipulations concerning relations between sets of units of meaning are extended to terms and simple sentences. Thus, in the case of strong entailment the conclusion should not contain simple sentences which are lacking in the premisses. From this point of view

$$(\forall s^1)(s^1 \leftarrow P) \vdash (s^2 \leftarrow P)$$
$$(s^2 \leftarrow P) \vdash (\exists s^1)(s^1 \leftarrow P)$$

cannot be accepted as rules of logical entailment.

12. INTRODUCTION AND ELIMINATION OF QUANTIFIERS

A1. If t does not occur in X, then

$$(\alpha \reflectbox{K} t) X \dashv\vdash X.$$

We assume the assertions

A2. $X \vdash (\exists t)\, X$
A3. $(\forall t)\, X \vdash X$
A4. $(X \vdash Y) \rightarrow ((>\!\!|\, t)\, X \vdash (>\!\!|\, t)\, Y)$
$(X \vdash Y) \rightarrow ((\neg >\!\!|\, t)\, Y \succ\!\!- (\neg >\!\!|\, t)\, X) \cdot ((?>\!\!|\, t)\, Y \succ\!\!- (?>\!\!|\, t)\, X)$,

where $>\!\!|$ is either of $\forall$ and $\exists$.

The consequences of *A3* are:

T2. $(\forall s)(s\alpha \leftarrow P) \dashv\vdash (\neg \exists s)(s\beta \leftarrow P) \cdot (\neg \exists s)(s\gamma \leftarrow P)$
$(\exists s)(s\alpha \leftarrow P) \dashv\vdash (\neg \forall s)(s\beta \leftarrow P) \cdot (\neg \forall s)(s\gamma \leftarrow P)$
$(\neg \forall s)(s\alpha \leftarrow P) \dashv\vdash (\exists s)(s\beta \leftarrow P) \vee (\exists s)(s\gamma \leftarrow P)$
$(?\forall s)(s\alpha \leftarrow P) \dashv\vdash (?\exists s)(s\beta \leftarrow P) \vee (?\exists s)(s\gamma \leftarrow P)$.

A5. If $\vdash X$, then $\vdash (\forall t) X$.
A6. If t is an individual term, then

$$(\forall t)\, X \dashv\vdash X.$$

The other cases of introduction and elimination of quantifiers will be taken up in connection with the account of sentential structure.

13. QUANTIFIERS AND THE SIGNS "AND" AND "OR"

Let $X^1, X^2, \ldots, X^n$ be any sentences. Thanks to *A4* of the previous section we obtain:

T1. $(\exists t)(X^1 \cdot \ldots \cdot X^n) \vdash (\exists t)\, X^1 \cdot \ldots \cdot (\exists t)\, X^n$
T2. $(\forall t)(X^1 \cdot \ldots \cdot X^n) \vdash (\forall t)\, X^1 \cdot \ldots \cdot (\forall t)\, X^n$
T3. $(\neg \exists t)\, X^1 \;|\!\succ\!\!- (\neg \exists t)(X^1 \cdot \ldots \cdot X^n)$
T4. $(\neg \forall t)\, X^1 \;|\!\succ\!\!- (\neg \forall t)(X^1 \cdot \ldots \cdot X^n)$,

etc. But these do not exhaust the possible cases.

We assume the assertions *A1*:

a) $(\exists t)\, X^1 \cdot \ldots \cdot (\exists t)\, X^n \vdash (\exists t)(X^1 \vee \ldots \vee X^n)$
b) $(\forall t)\, X^1 \cdot (\exists t)\, X^2 \vdash (\exists t)(X^1 \cdot X^2)$
$(\forall t)\, X^1 \cdot \ldots \cdot (\forall t)\, X^{n-1} \cdot (\exists t)\, X^n \vdash (\exists t)(X^1 \cdot X^2 \cdot \ldots \cdot X^n)$
c) $(\neg \exists t)(X^1 \cdot \ldots \cdot X^n) \dashv\vdash (\neg \exists t)\, X^1 \vee \ldots \vee (\neg \exists t)\, X^n$
d) $(?\exists t)(X^1 \cdot \ldots \cdot X^n) \dashv\vdash ((?\exists t)\, X^1 \vee \ldots \vee (?\exists t)\, X^n) \cdot$
$\cdot \sim ((\neg \exists t)\, X^1) \cdot \ldots \cdot \sim ((\neg \exists t)\, X^n)$.

The assertions *A2*:

a) $(\forall t)X^1\cdot\ldots\cdot(\forall t)X^n \vdash (\forall t)(X^1\cdot\ldots\cdot X^n)$
b) $(\neg\forall t)(X^1\cdot\ldots\cdot X^n) \dashv\vdash (\neg\forall t)X^1 \vee\ldots\vee (\neg\forall t)X^n$
c) $(?\forall t)(X^1\cdot\ldots\cdot X^n) \dashv\vdash ((?\forall t)X^1 \vee\ldots\vee (?\forall t)X^n)\cdot$
$\cdot\sim((\neg\forall t)X^1)\cdot\ldots\cdot\sim((\neg\forall t)X^n)$.

Let it be the case that X^{1*} is $(X^1\sim X^2\cdot\ldots\cdot\sim X^n)$, X^{2*} is $(X^2\sim X^1\cdot\ldots\cdot\sim X^n)$,..., X^{n*} is $(X^n\sim X^1\cdot\ldots\cdot\sim X^{n-1})$; Z^1 is $(\neg\exists t)(X^1X^2\cdot\ldots\cdot X^n)$, Z^2 is $(\neg\exists t)(\sim X^1\sim X^2\cdot\ldots\cdot\sim X^n)$; Y^1 is $(\exists t)X^{1*}$, Y^2 is $(\exists t)X^{2*}$,..., Y^n is $(\exists t)X^{n*}$; Y^{1*} is $(\neg\exists t)X^1$, Y^{2*} is $(\neg\exists t)X^2$,..., Y^{n*} is $(\neg\exists t)X^n$; W is $(Z^1Z^2Y^1\cdot\ldots\cdot Y^n)$; W^1,..., W^k are all possible sentences which are formed from W through replacement of one or more (but not more than $n-2$) of Y^1,..., Y^n by corresponding sentences from among Y^{1*},..., Y^{n*}; V^1,..., V^m are all possible sentences formed from $(X^1X^2\cdot\ldots\cdot X^n)$ through replacement of all or i (where $1\leqslant i\leqslant n-2$) sentences of the number X^1, X^2,..., X^n by corresponding sentences from among $\sim X^1$, $\sim X^2$,..., $\sim X^n$.

The assertions *A3*:

a) $(\forall t)(X^1:X^2) \dashv\vdash (\forall t)(X^1\sim X^2):(\forall t)(\sim X^1X^2):(\exists t)$
$(X^1\sim X^2)(\exists t)(\sim X^1X^2)(\neg\exists t)(X^1X^2)(\neg\exists t)$
$(\sim X^1\sim X^2)$; $(\forall t)(X^1:X^2:\ldots:X^n) \dashv\vdash (\forall t)X^{1*}:$
$:(\forall t)X^{2*}:\ldots:(\forall t)X^{n*}:W:W^1:\ldots:W^k$
b) $(\neg\forall t)(X^1:X^2) \dashv\vdash (\exists t)(X^1X^2) \vee (\exists t)(\sim X^1\sim X^2)$
$(\neg\forall t)(X^1:X^2:\ldots:X^n) \dashv\vdash (\exists t)(X^1X^2\cdot\ldots\cdot X^n) \vee$
$\ldots\vee(\exists t)V^1 \vee\ldots\vee (\exists t)V^m$
c) $(?\forall t)(X^1:X^2) \dashv\vdash ((?\exists t)(X^1X^2) \vee (?\exists t)(\sim X^1\cdot\sim X^2))\sim$
$\sim((\neg\exists t)(X^1X^2))\sim((\neg\exists t)(\sim X^1\sim X^2))$; $(?\forall t)(X^1:\ldots:X^n) \dashv\vdash$
$\dashv\vdash ((?\exists t)(X^1X^2\cdot\ldots\cdot X^n) \vee (?\exists t)V^1 \vee (?\exists t)V^m)\sim$
$\sim((\neg\exists t)(X^1X^2\cdot\ldots\cdot X^n)\cdot\sim((\neg\exists t)V^1)\cdot\ldots\cdot\sim((\neg\exists t)V^m)$.

Let it be the case that F^1 is $(\exists t)(X^1\cdot\ldots\cdot X^n)$, F^2 is $(\exists t)\,V^1$,..., F^{m+1} is $(\exists t)\,V^m$; F^{1*} is $(\neg\exists t)(X^1\cdot\ldots\cdot X^n)$, F^{2*} is $(\neg\exists t)\,V^1$,..., $F^{(m+1)*}$ is $(\neg\exists t)\,V^m$; F_1 is $(\neg\exists t)(X^1\sim X^2\cdot\ldots\cdot\sim X^n)$, F_2 is $(\neg\exists t)(X^2\sim X^1\cdot\ldots\cdot\sim\sim X^n)$,..., F_n is $(\neg\exists t)(X^n\sim X^1\cdot\ldots\cdot\sim X^{n-1})$; Q is $(F_1F_2\cdot\ldots\cdot F^{m+1}F_1F_2\cdot\ldots\cdot F_n)$; Q^1,..., Q^l are all possible sentences which are formed from Q by replacing i $(1\leqslant i\leqslant m+1)$ sentences from among F^1,..., F^{m+1} by corresponding sentences from among F^{1*},..., $F^{(m+1)*}$.

The assertions *A4*:

a) $(\exists t)(X^1{:}X^2)\dashv\vdash(\neg\forall t)(X^1X^2)(\neg\forall t)(\sim X^1\sim X^2);(\exists t)$
$(X^1{:}X^2{:}\ldots{:}X^n)\dashv\vdash(\neg\forall t)(X^1X^2\cdot\ldots\cdot X^n)\cdot(\neg\forall t)\,V^1\cdot\ldots\cdot(\neg\forall t)\,V^n$
b) $(\neg\exists t)(X^1{:}X^2)\dashv\vdash(\forall t)(X^1X^2){:}(\forall t)(\sim X^1\sim X^2){:}$
${:}(\exists t)(X^1X^2)(\exists t)(\sim X^1\sim X^2)(\neg\exists t)(X^1\sim X^2)\cdot$
$\cdot(\neg\exists t)(\sim X^1X^2);(\neg\exists t)(X^1{:}X^2{:}\ldots{:}X^n)\dashv\vdash(\forall t)$
$(X^1X^2\cdot\ldots\cdot X^n){:}(\forall t)\,V^1{:}\ldots{:}(\forall t)\,V^m{:}Q{:}Q^1{:}\ldots{:}Q^l$
c) $(?\exists t)(X^1{:}X^2)\dashv\vdash(?\forall t)(X^1X^2)(?\forall t)(\sim X^1\sim X^2);(?\exists t)$
$(X^1{:}X^2{:}\ldots{:}X^n)\dashv\vdash(?\forall t)(X^1X^2\cdot\ldots\cdot X^n)\cdot(?\forall t)\,V^1\cdot\ldots\cdot(?\forall t)\,V^n.$

The assertions *A5*:

a) $(\forall t)\,X^1{:}\ldots{:}(\forall t)\,X^n\vdash(\forall t)(X^1{:}\ldots{:}X^n)$
b) $(\forall t)\,X^1{:}(\exists t)\,X^2\vdash(\exists t)(X^1{:}X^2)$
$(\forall t)\,X^1{:}\ldots{:}(\forall t)\,X^{n-1}{:}(\exists t)\,X^n\vdash(\exists t)(X^1{:}\ldots{:}X^n)$
c) $(\exists t)\,X^1{:}\ldots{:}(\exists t)\,X^n\vdash(\exists t)(X^1{:}\ldots{:}X^n)$
d) $(\forall t)\,X^1{:}(?\forall t)\,X^2\vdash(?\forall t)(X^1{:}X^2)$
$(\alpha^1\forall t)\,X^1{:}\ldots{:}(\alpha^n\forall t)\,X^n\vdash(?\forall t)(X^1{:}\ldots{:}X^n),$

where $\alpha^1,\ldots,\alpha^n$ indicate that in at least one of $\alpha^1\forall t,\ldots,\alpha^n\forall t$ there is a ? before $\forall$;

e) $(\exists t)\,X^1{:}(?\exists t)\,X^2\vdash(?\exists t)(X^1{:}X^2)$
$(\alpha^1\exists t)\,X^1{:}\ldots{:}(\alpha^n\exists t)\,X^n\vdash(?\exists t)(X^1{:}\ldots{:}X^n),$

where $\alpha^1,\ldots,\alpha^n$ are the same as above.

Consequences of *A1–A5* for the sign $\vee$:

a) $(\exists t)(X^1\vee\ldots\vee X^n)\dashv\vdash\exists(t)\,X^1\vee\ldots\vee(\exists t)\,X^n$
b) $(\neg\exists t)(X^1\vee\ldots\vee X^n)\dashv\vdash(\forall t)(\sim X^1\cdot\ldots\cdot\sim X^n)$
$(\neg\forall t)(X^1\vee\ldots\vee X^n)\dashv\vdash(\exists t)(\sim X^1\cdot\ldots\cdot\sim X^n)$
c) $(?\exists t)(X^1\vee\ldots\vee X^n)\dashv\vdash(?\forall t)(\sim X^1\cdot\ldots\cdot\sim X^n)$
$(?\forall t)(X^1\vee\ldots\vee X^n)\dashv\vdash(?\exists t)(\sim X^1\cdot\ldots\cdot\sim X^n)$

14. SYLLOGISTICS OF PROPERTIES

We have already noted that the word "is" is not known for its clarity and is not unambiguous. The formation of sentences of the form "*s* is *P*" in the construction of syllogistics, as is usually done, leaves unspecified just which sentences are intended: sentences on the belonging of attributes, on inclusion in a class, or on the identity or inclusion of terms according to value. And these are not the same.

One must distinguish the syllogistics of classes from the syllogistics of properties (attributes). A syllogistic of properties is formed by the following assertions (and consequences inferred from them):

A1. $(\exists s)(s\alpha \leftarrow P) \dashv\vdash (\exists s^*\alpha \downarrow P)(s^*\alpha \downarrow P \leftarrow P^* \downarrow s)$
A2. $(\forall s)(s\alpha \leftarrow P) \dashv\vdash (\forall s^*\beta \downarrow P)(\forall s^*\gamma \downarrow P)(\exists P^* \downarrow s)$
$((s^*\beta \downarrow P \urcorner \leftarrow P^* \downarrow s)(s^*\gamma \downarrow P \urcorner \leftarrow P^* \downarrow s))$
A3. $(\reflectbox{$\lambda$} s)(s\alpha \leftarrow P^1)(\forall s^*\alpha \downarrow P^1)((s^*\alpha \downarrow P^1)\,\beta \leftarrow P^2) \vdash (\reflectbox{λ} s)(s\beta \leftarrow P^2)$.

In the classical case *A1* and *A2* have the form

A^c1. $(\exists s)(s \leftarrow P) \dashv\vdash (\exists sc \downarrow P)(sc \downarrow P \leftarrow Pc \downarrow s)$
A^c2. $(\forall s)(s \leftarrow P) \dashv\vdash (\forall sc \sim\downarrow P) \sim (sc \sim\downarrow P \leftarrow Pc \downarrow s)$.

15. IMPLICIT QUANTIFIERS

The assertion

$$(t^1 \rightharpoonup t^2)\cdot(t^1 \leftarrow PX) \rightarrow (t^2 \leftarrow PX(t^1/t^2))$$

(i.e. "That which is true of the genus is true of the species") is not always valid. For example, it is possible that $s^1 \rightharpoonup s^2, s^1 \leftarrow P$ be true for some individual of the value-range of s^1, and $s^2 \leftarrow P$ be not true for some individual of the value-range of s^2 (for another individual of the value-range of s^1).

If the adduced assertion is assumed then in fact the universal quantifier ("all") is implicitly presupposed in the sentence $t^1 \leftarrow PX$, i.e. the latter is used as $(\forall t^1)(t^1 \leftarrow PX)$. For example, when we assert "The sum of the angles of a triangle is equal to 180°" we have in mind (we assume) all triangles (each, any, every triangle). Such a use of a sentence usually happens when X results from a definition of t^1. In the case of abstract objects all true sentences which contain t^1 are of this type.

Of interest is the case of implicit quantification

$$(s^1 \downarrow PX \rightharpoonup s^2) \rightarrow (\forall s^2)(s^2 \leftarrow PX(s^1/s^2)),$$

where s^1 is free in X.

16. TERMS

We obtain terms from quantified sentences according to the schema

$$t \downarrow PX,$$

where t is free in X ("t is such that X"). In particular the term

$$P^i \downarrow P((\forall s)(s \leftarrow P^i)),$$

which reads "P^i such that it is present to all s", is important. It will be abbreviated as

$$P^i \downarrow \forall s.$$

The assertion

$$A1.\ (s^1 \rightharpoonup s^2) \leftrightarrow (\forall s^2)(\forall P^* \downarrow \forall s^1)(s^2 \leftarrow P^* \downarrow \forall s^1)$$

holds for this term.

17. PARTIAL QUANTIFICATION

In the case of complex terms it is sometimes enough to quantify their parts. Thus, the following assertions are valid:

$$(\forall s)\, X^1 \vdash (\forall s \downarrow X^2)\, X^1$$
$$(\exists s \downarrow X^1)\, X^2 \vdash (\exists s)\, X^2$$
$$(\forall P^1)(s \downarrow P^1 \leftarrow P^2) \rightarrow (\forall P^1)((s \leftarrow P^1) \rightarrow (s \leftarrow P^2)).$$

18. CONSTRUCTION OF SENTENCES

From the point of view of construction quantified sentences can be divided into two groups. The first group includes those sentences, for the obtaining of which from other given sentences one need only know the rules of logical entailment. The second group includes the rest. It is known that a significant portion of quantified sentences are obtained in such a way that only some rules of logical entailment are insufficient for their "justification". Since the problem of the construction of such sentences is non-trivial only for sentences with the universal quantifier, this can be called the problem of generalization.

There are several paths of generalization. We will mention some of them and we will formulate some of their principles as an illustration of what can be categorically said in this regard by the logical theory of scientific knowledge and of possible directions for applying its efforts.

The simplest case of complete induction consists in the following. Let X be a sentence in which s occurs. Let $s^1,\dots, s^n$ $(n \geqslant 1)$ be all the individuals of the value-range of s and n be a finite number. If the sentence

$$(s^1 \leftarrow PX(s/s^1))\cdot\ldots\cdot(s^n \leftarrow PX(s/s^n))$$

is true, then by the definition of $(\forall s)X$ and $[(\forall s)X] \leftarrow v^1$, $(\forall s)X$ will also be true.

A more complex case is the obtaining of $(\forall s)X$ from the sentences

$$(\forall s_1)\, X(s/s_1),\dots, (\forall s_m)\, X(s/s_m),$$

where $s_1,\dots, s_m$ (m is a finite number) form a partition of s. The universal sentence is here obtained precisely by virtue of definitions.

In both of these cases an extra-logical hypothesis is used in obtaining the universal sentence:

1) in the first case it is the assumption that $s^1,\dots, s^n$ are all individuals of the value-range of s;

2) in the second case it is the assumption that $s_1,\dots, s_m$ form a partition of s, i.e., the individuals of the value-range of $s_1,\dots, s_m$ exhaust the set of individuals of the value-range of s.

Let there be a definition of s such that:

1) $s \rightharpoonup s^1,\dots, s \rightharpoonup s^n$ (n is finite);

2) if $s \rightharpoonup s_1,\dots, s \rightharpoonup s_m$, then $s \rightharpoonup s^{i1},\dots, s \rightharpoonup s^{ik}$, where m and k are finite;

3) there are no other s's.

Let

$$(\forall s^1)\,(s^1 \leftarrow PX(s/s^1)),\dots, (\forall s^n)\,(s^n \leftarrow PX(s/s^n)),$$

be true and let it be established in some way (in particular as proved) that

$$\begin{aligned}&(\forall s_1)\,(s_1 \leftarrow PX(s/s_1))\cdot\ldots\cdot(\forall s_m)\,(s_m \leftarrow PX(s/s_m)) \rightarrow\\ &\rightarrow (\forall s^{i1})\,(s^{i1} \leftarrow PX(s/s^{i1}))\cdot\ldots\cdot(\forall s^{ik})\,(s^{ik} \leftarrow PX(s/s^{ik})).\end{aligned}$$

is true.

Then

$$\begin{aligned}&(\forall s^1)\,(s^1 \leftarrow PX(s/s^1))\cdot\ldots\cdot(\forall s^n)\,(s^n \leftarrow PX(s/s^n))\cdot\\ &\cdot(\forall s^{i1})\,(s^{i1} \leftarrow PX(s/s^{i1}))\cdot\ldots\cdot(\forall s^{ik})\,(s^{ik} \leftarrow PX(s/s^{ik}))\end{aligned}$$

will be true and

$$(\forall s)\, X$$

by definition (since there are no other s's).

The extra-logical assumption here is the method of enumerating all the individuals of s, presupposed in the definition of the latter. Moreover, the class of individuals here can be infinite.

In the case of mathematical induction one assumes (presupposes or discovers in the properties of the objects) the possibility of ordering the individuals of s and of constructing the assertion

$$(s^n \leftarrow PX(s/s^n)) \rightarrow (s^{n+1} \leftarrow PX(s/s^{n+1})),$$

where s^n is any individual. If

$$s^1 \leftarrow PX(s/s^1)$$

is true and the previous assertion is true, then

$$(\forall s)\, X.$$

is true.

But here again one uses extra-logical assumptions.

Complete induction concerning empirical objects is trivial and possible only for a finite (and even, practically observable) number of individuals of a given class. In the case of abstract objects it is a very effective method (also for infinite classes) which is extensively used in logic, mathematics and other "mathematized" sciences.

Complete induction can also be called conclusive, strict or necessary.

If the number of individuals of a given class is infinite or such that it is practically impossible to examine all of them (or for any reason it is impossible to examine all the individuals of this class) and the use of the methods of complete induction is excluded, then one uses the so-called partial, empirical or probabilistic induction.

There are different forms of partial induction, each of which is based on certain principles (assumptions). The latter are usually not explicitly formulated. But once there is partial induction these principles are a fact. We will look at some types of partial induction.

Quantitative induction:

1) if the number of cases where $s \leftarrow PX$ is large enough and there are

no cases where $s \leftarrow P \sim X$, then $(\forall s)X$ is considered true (popular induction);

2) if the probability of $s \leftarrow PX$ is high enough, then $(\forall s)X$ is considered true (frequency induction).

But when the "enough" is the case depends on circumstances. No logical criteria are formulated here. Experience and luck play the key role. It can happen that the investigator "stumbles" as it were across an X such that $(\forall s)X$ even though he has examined only a few samples of s. But it can also happen that the investigator examines a great number of s, constructs $(\forall s)X$, and then comes upon an s such that $s \leftarrow P \sim X$. What is more, there are cases where one knows immediately that $s \leftarrow P \sim X$ is possible but operates with $(\forall s)X$ as if it were true.

Conditional induction: if $s \leftarrow PX$ in certain circumstances, then $(\forall s)X$ is considered true under these circumstances. The effect here depends on the fullness, exactness, etc., of the enumeration of the conditions. A rather clear principle is possible here: "If $s \leftarrow PX$ is true then it is possible to establish (to fix) conditions such that under these conditions $s \leftarrow PX$ is always true, i.e., $(\forall s)X$". The principle is in theory irreproachable. But in practical use its effect depends on circumstances. Thus the sentence "A man can become Emperor of France" is true of Napoleon I. One can list the conditions necessary for this; and under these conditions this sentence will be true for all people; but these conditions are not always met and not by all people. In the practical application of this principle common sense always plays a role, limiting the character of X and the description of the conditions under which $s \leftarrow PX$.

Conditional-quantitative induction: one chooses any objects s (at least two); if with a sufficiently large number of cases and sufficient variety of conditions (the limit case is mutual exclusion) $s \leftarrow PX$ is true, then $(\forall s)X$ is considered true.

Induction through difference: if individuals $s^1, \ldots, s^n$ of a class s are sufficiently different and $s^1 \leftarrow PX(s/s^1), \ldots, s^n \leftarrow PX(s/s^n)$ are true then $(\forall s)X$ is considered true. Induction through similarity: if $s^1 \leftarrow PX(s/s^1), \ldots, s^n \leftarrow PX(s/s^n)$ are true, all individuals $s^1, \ldots, s^n$ are sufficiently similar, and Ks includes only $s^1, \ldots, s^n$ and such individuals as are sufficiently similar to them, then $(\forall s)X$ is considered true.

In the above formulae we used expressions which have to be clarified and which can be clarified within the framework of logic. No matter how clear the concepts are, however, the fact remains that some extra-logical (heuristic) assumptions are necessary.

One also employs the following operation (reduction): 1) one assumes $(\alpha\text{ꓘ}s)X$; 2) from it (and from other sentences which are considered true) conclusions are drawn; 3) if these conclusions are true, if their number and importance are sufficiently great, then $(\alpha\text{ꓘ}s)X$ is taken as true. It is clear that this "sufficiently great" has an extra-logical nature, depends on circumstances, is subject to variations, etc. In the limit case the conclusions are sharply defined and the possibility of obtaining them with the help of $(\alpha\text{ꓘ}s)X$ is sufficient to recognize the latter as true.

Two variants of reduction are possible. Strong variant: if from $(\alpha\text{ꓘ}s)X$ we get at least one conclusion which is not true then $(\alpha\text{ꓘ}s)X$ is not true. Weak variant: conclusions which are not true may be obtained from $(\alpha\text{ꓘ}s)X$; but if these do not play an important role (or can be eliminated), then $(\alpha\text{ꓘ}s)X$ can be taken as true. Here there is a question about the "weight" (importance) of the conclusion. If the "weight" of true conclusions from $(\alpha\text{ꓘ}s)X$ is rated as α and of the not-true ones as β, then the relation between α and β determines whether or not one will decide to consider it true.

There are no logical rules which would guarantee in all cases the obtaining of true sentences like

$$(\forall s)\,X$$

from sentences like

$$s^i \leftarrow PX(s/s^i)\,.$$

Efforts to discover or contrive them are automatically doomed.

19. DEFINITIONS AND ASSERTIONS

Let

$$s^2 = Df.\, s^1 \downarrow X$$

be a definition. And let X be true relative to at least one individual of the value-range of s^1, i.e.,

$$(s^1 \downarrow X) \leftarrow E\,.$$

There is the rule: if $(s^1 \downarrow X) \leftarrow E$ and $s^2 = Df.\, s^1 \downarrow X$, then

$$(\forall s^2)\, X(s^1/s^2).$$

Since the definition is accepted and the indicated individual exists, then the general sentence above is true.

For example, let there exist an s^1 such that $s^1 \leftarrow P$ is true and

$$s^2 = Df.\, s^1 \downarrow P.$$

There is the rule: if $(s^1 \downarrow P) \leftarrow E$ and $s^2 = Df.\, s^1 \downarrow P$, then $(\forall s^2)(s^2 \leftarrow P)$. Whence we find that $(\forall s^2)(s^2 \leftarrow P)$ is true.

General sentences are often obtained in science in this way (*viz.* by definitional generalization). Such are the sentences "All electrons are negatively charged", "All protons are positively charged", etc., because reference to the negative charge occurs in the definitions of the term "electron" and reference to the positive charge occurs in the definition of the term "proton", etc. And just as often the nature of such generalizations is forgotten. An important segment of scientific discussions arise precisely because scientists depart in fact (but not explicitly) from previously agreed stipulations and accept new ones, considering their consequences to be contradictory hypotheses or empirical truths.

20. CLASSICAL AND NON-CLASSICAL RELATIONS BETWEEN SENTENCES

Assuming indeterminacy and distinguishing negations in complex logic, we studied the non-classical case of relations between sentences in the following sense. If we introduce brackets for the predicates then we can also write it with the help of one negation:

1) $\vdash ((P)(s) : (\sim P)(s) : \sim(P)(s) \cdot \sim(\sim P)(s))$
2) $\vdash ((\forall t)\, X : (\sim \forall t)\, X : \sim(\forall t)\, X \cdot \sim(\sim \forall t)\, X)$
3) $\vdash ((\exists t)\, X : (\sim \exists t)\, X : \sim(\exists t)\, X \cdot \sim(\sim \exists t)\, X)$

It now remains that we accept the formulae $(\sim P)(s) \dashv\vdash \sim(P)(s)$, $(\sim \forall t) X \dashv\vdash \sim(\forall t) X$ and $(\sim \exists t) X \dashv\vdash \sim(\exists t) X$ and we obtain the classical case.

Such a method of writing is useful since it makes clear the following assertion: for the non-classical cases considered in complex logic all other logically conceivable possibilities are excluded. Let us take, for example, $(P)(s)$ and $(\sim P)(s)$. The third possibility, $\sim(P)(s) \sim (\sim P)(s)$, is fully admissible if $\sim(\sim P)(s) \vdash (P)(s)$ is not provable. But the fourth possibility is excluded. Similarly for quantifiers.

If we assume that $\vdash P(s) : \neg P(s) : ?P(s) : \sim P(s) \sim \neg P(s) \sim ?P(s)$ is provable, then $?P(s)$ cannot be presented as $\sim P(s) \sim \neg P(s)$, but we have to find another interpretation for indeterminacy. We also would have to find an interpretation for $\sim P(s) \sim \neg P(s) \sim ?P(s)$ but not through classical negation. We do not know of such cases.

CHAPTER ELEVEN

THEORY OF QUANTIFIERS

1. PARADOXES OF THEORY OF QUANTIFIERS

In classical logic of predicates

$$(\forall s^1)\, P(s^1) \supset P(s^2)$$
$$P(s^2) \supset (\exists s^1)\, P(s^1)$$

and other formulae are generally valid and, of course, provable, where the consequent contains individual variables which are lacking in the antecedent. This is the case because the classical logic of predicates is based on the assumption: the value-ranges of all individual variables are identical.

But if we take the general case and assume that the value-ranges of the individual variables occurring in a formula could not be identical, then the formulae in question are not generally valid and one cannot take them as rules of logical entailment. In another form, not all the truth conditions of the premisses in the formulae

$$(\forall s^1)\, P(s^1) \vdash P(s^2)$$
$$P(s^2) \vdash (\exists s^1)\, P(s^1)$$

are truth conditions of the conclusions if it is possible that the value-ranges of s^1 and s^2 are not identical. Something similar is the case for other formulae where the conclusion contains individual variables which are lacking in the premisses. We will call such formulae intuitively paradoxical.

We will call intuitively non-paradoxical a theory of logical entailment for quantifiers, the proper part of which has the following property. In a system obtained by joining it with S^1 (or S_1) all provable formulae of the form $x \vdash y$ are such that in y there are no occurrences of formulae of the type $Q(a)$ which are lacking in x. This calculus will have to be the basis of any theory of quantifiers. This chapter is devoted to the discussion of the problem of constructing such a calculus.

2. CLASSICAL AND NON-CLASSICAL CASES

Below we will use the expressions "classical case of the theory of quantifiers" (or "classical theory of quantifiers") and "non-classical case of the theory of quantifiers" (or "non-classical theory of quantifiers"). They are here distinct from the oft-used terms "classical logic (calculus) of predicates" and "non-classical logic (calculus) of predicates". By non-classical theory of quantifiers we mean one which includes the possibility of indeterminacy of quantifiers and of a distinction between two negations; by classical we mean one which excludes indeterminacy and distinction of negations. Classical and non-classical calculi of predicates belong to the classical case of the theory of quantifiers in our use of the terms.

3. RESTRICTION OF THE CLASSICAL CALCULUS OF PREDICATES

It first has to be said that the classical case of the strong theory of quantifiers can be obtained through a certain restriction of and change in the classical predicate calculus with material implication and the axioms A

$$(\forall s^1)\, P(s^1) \supset P(s^2)$$
$$P(s^2) \supset (\exists s^1)\, P(s^1),$$

which are the only ones which allow of the substitution of individual variables in appropriate formulae of the type $Q(a)$. This restriction and change is as follows:

1) instead of classical sentential calculus, we take S^1 (or S_1);

2) in the remaining axioms and rules of inference the principal sign of material implication is replaced by the sign of logical entailment;

3) the axioms A are replaced by the axioms B

$$(\forall s)\, P(s) \vdash P(s)$$
$$P(s) \vdash (\exists s)\, P(s);$$

4) the transformation rule for bound variables is restricted so that its application does not lead to the occurrence in the conclusion of variables lacking in the premisses.

A calculus obtained in this way is non-paradoxical: formulae of the type $Q(a)$ which are lacking in the premisses cannot appear in the conclusions of provable formulae $x \vdash y$. Obviously, all the $x \vdash y$ formulae provable in it are generally valid. And all generally valid formulae $x \vdash y$, in which there is no occurrence in y of formulae like $Q(a)$ lacking in x, are provable in it.

If instead of S^1 (or S_1) we take other systems S^i (or S_i) then we obtain the classical case of weakened, maximal and converse theory of quantifiers. The sentential variables or formulae of the type $Q(a)$ have to be taken into account in the transitivity rule and in the axiomatic schema. For degenerate entailment one needs a generalization rule: if $\vdash X$, then $(\forall s)X$.

However, below we will construct a theory of quantifiers which will be as close as possible to the intuitive base, from which we deviated above.

4. CLASSICAL STRONG THEORY OF QUANTIFIERS

The system S_c^1 of classical strong theory of quantifiers is formed by means of the following additions to S_1 (or S^1) and modifications of it.

Alphabet:

1) $s, s^1, s^2, \ldots$ are subject variables (individual variables);

2) $P, P^1, P^2, \ldots$ are predicate variables;

3) $is, is^1, is^2, \ldots, iP, iP^1, iP^2, \ldots$ are designated subject and predicate variables;

4) $\forall$, $\exists$ are quantifiers.

D1. The group of subject variables:

1) variables and designated subject variables are groups of subject variables;

2) if $a^1, \ldots, a^n$ $(n \geqslant 2)$ are groups of subject variables, then $(a^1, \ldots, a^n)$ is a group of subject variables;

3) something is a group of subject variables only by virtue of 1 and 2.

D2. K-formula:

1) if Q is a predicate variable or a designated predicate variable, and a is a group of subject variables, then $Q(a)$ is a K-formula;

2) if X is a K-formula and a is a group of subject variables or a predicate variable, then $\sim X$, $(\forall a)X$ and $(\exists a)X$ are K-formulae;

3) if $X^1, \ldots, X^n$ $(n \geqslant 2)$ are K-formulae, then $X^1 \cdot \ldots \cdot X^n$ and $X^1 : \ldots : X^n$ (or $X^1 \vee \ldots \vee X^n$ depending on the general theory of deduction involved) are K-formulae;

4) something is a K-formula only by virtue of 1)–3).

D3. The K-formulae mentioned in point 2 of definition *D2* are elementary K-formulae;

D4. Addition to the definition of occurrence: a occurs in $(\forall a)$, $(\exists a)$, $Q(a)$; $a^i (i=1,\ldots,n)$ occurs in $(a^1,\ldots,a^n)$; $(\forall a)$ and $(\exists a)$ occur, respectively, in $(\forall a)X$ and $(\exists a)X$; Q occurs in $Q(a)$; X occurs in $(\forall a)X$ and $(\exists a)X$; if a occurs in X and X occurs in Y, then a occurs in Y.

D5. If a group of subject or predicate variables a occurs in X and $(\forall a)$ and $(\exists a)X$ do not occur in X, then a occurs freely in X (a is not bound in X). If a occurs in X then a occurs bound (a is bound and not free) in $(\forall a)X$ and $(\forall a)X$. If X occurs in Y and a is free (bound) in X, then the occurrence of a in X is a free (bound) occurrence of a in Y.

D6. In the definition of $X \vdash Y$, X and Y are K-formulae.

D7. $X \vdash Y$ is a proper formula of logical entailment if and only if X and Y are proper K-formulae.

D8. $(\forall a)$ and $(\exists a)$ are degenerate quantifiers in $(\forall a)X$ and $(\exists a)X$, respectively, if and only if a does not occur freely in X and X is a proper K-formula.

In what follows we shall use symbols like

$$Xia$$

in the following sense: if X is a K-formula, then Xia is a K-formula which is formed from X by putting ia for a wherever a freely occurs in X, and a is a subject or predicate variable.

D9. Intuitive interpretation of K-formulae:

1) $X1a \cdot \ldots \cdot Xna (n \geqslant 1)$ is an intuitive interpretation of $(\forall a)X$ with respect to a;

1) $X1a \vee \ldots \vee Xna$ is an intuitive interpretation of $(\exists a)X$ with respect to a.

D10. A complete intuitive interpretation of a K-formula is its interpretation with respect to all bound variables occurring in it.

D11. The intuitive interpretation of the formula $X \vdash Y$ is formed as follows:

1) if a subject or predicate variable freely occurs in X or Y then $X \vdash Y$ is replaced by $(\forall a)X \vdash (\forall a)Y$ or $(\exists a)X \vdash (\exists a)X$; and so on for all free variables occurring in X and Y;

2) all degenerate quantifiers are rejected from the resulting formulae;

3) in the formula resulting from the first two steps the premisses and conclusion are replaced by their complete intuitive interpretations.

Semantic interpretation of formulae:

1) the elementary K-formula takes the values v^1 and nv^1 (as in the case of sentential variables above); and two elementary K-formulae are different if and only if they are written differently;

2) if the symbol X is a K-formula, then the symbol Xia takes the value 1 or 0 independently of X and also as an elementary K-formula;

3) if a does not occur in X then $Xia \approx X$ (i.e., the values of Xia and X are identical);

4) $Xiajb \approx Xjbia$

5) $(X^1 \cdot \ldots \cdot X^k)ia$, $(X^1 \vee \ldots \vee X^k)ia$, $({\sim} X)ia$ are, respectively, equivalent to $X^1ia \cdot \ldots \cdot X^kia$, $X^1ia \vee \ldots \vee X^kia$, ${\sim} Xia$.

D12. The formula $X \vdash Y$ is generally valid if and only if its intuitive interpretation is a tautology with any number of designated variables for every variable occurring in $X \vdash Y$.

D13. $(\forall (a^1, \ldots, a^n))X$ (where $n \geqslant 2$) is short for $(\forall a^1) \ldots (\forall a^n)X$; $(\exists (a^1, \ldots, a^n))X$ is short for $(\exists a^1) \ldots (\exists a^n)X$.

Axiomatic schemata of S_c^1:

1. $(\forall a) X \vdash X$
2. $X \vdash (\exists a) X$
3. $(\forall a) X \vdash {\sim} (\exists a) {\sim} X$
4. ${\sim} (\exists a) {\sim} X \vdash (\forall a) X$
5. $(\forall a) X (\exists a) Y \vdash (\exists a)(XY)$
6. $(\forall a)(X \vee Y) \vdash (\exists a) X \vee (\forall a) Y$
7. $(\exists a) X \vdash (\forall a) X$,

where a does not occur freely in X, or $\vdash X$ is provable.

The axiomatic schemata replace the axioms of S^1 (S_1) and the substitution rule drops.

R1. If $X \vdash Y$, then $(\forall a)X \vdash (\forall a)Y$

R2. If $X \vdash Y$, then $(\exists a)X \vdash (\exists a)Y$.

D14. $X \vdash Y$ is provable in S_c^1 if and only if it is an axiom or an axiomatic schema of S_c^1 or is obtained from provable formulae according to the inference rules (including the additional rules of S_c^1).

T1. If $X \vdash Y$ is provable in S_c^1, then it is generally valid.

The proof of *T1* is trivial: all formulae in the list of axiomatic schemata are generally valid; and the interpretation of $X \vdash Y$ is identical to that of the formulae $(\forall a)X \vdash (\forall a) Y$ and $(\exists a)X \vdash (\exists a) Y$.

T2. If $X \vdash Y$ is provable in S_c^1, then in Y there are no elementary K-formulae lacking in X. The theorem is evident from the form of the axiomatic schemata and the rules of inference of S_c^1 and the properties of S^1 (or S_1).

We here leave open the question as to whether or not all generally valid formulae $X \vdash Y$ in S_c^1, in which Y does not contain elementary K-formulae lacking in X, are provable. The problem of completeness of the systems of the theory of quantifiers has been investigated in our *Complex Logic* (Moscow, 1970).

Let us introduce the notion of intuitive completeness of a system in the theory of quantifiers. In the logical system which defines the properties of the quantifiers there have to be provable formulae which are interpreted as: 1) rules of introduction and elimination of quantifiers; 2) rules of permutation of quantifiers; 3) rules of substitution of quantifiers; 4) rules for insertion and removal of quantifiers from disjunction and conjunction; 5) rules for negation. And these rules all have to be formulated for any sentences, i.e., not depending on their concrete structure. From this point of view axiomatic schemata are more suitable than axioms.

There is no difficulty in seeing that our system contains all admissible rules of this type.

T3. In S_c^1 the formulae

$$(\forall a)X(\forall a)Y \dashv\vdash (\forall a)(XY)$$
$$(\exists a)X \vee (\exists a)Y \dashv\vdash (\exists a)(X \vee Y)$$
$$(\exists a)(XY) \vdash (\exists a)X(\exists a)Y$$
$$(\forall a)X \vee (\forall a)Y \vdash (\forall a)(X \vee Y)$$

are provable. Similarly for any number of members of conjunction and disjunction.

$$(\exists a)X(\exists a)Y \vdash (\exists a)(XY)$$
$$(\forall a)(X \vee Y) \vdash (\forall a)X \vee (\forall a)Y$$

are excluded since the intuitive interpretations of these formulae are not generally valid.

T4. In S_c^1 the formulae

$$(\forall a) \sim X \dashv\vdash \sim (\exists a)X$$
$$(\exists a) \sim X \dashv\vdash \sim (\forall a)X$$
$$(\exists a)X \dashv\vdash \sim (\forall a) \sim X.$$

are provable. And because of the properties of S^1 (or S_1) the question on the introduction or elimination of negation is solved exhaustively.

T5. In S_c^1 the formulae

$$(\forall a)(\forall b)X \vdash (\forall b)(\forall a)X$$
$$(\exists a)(\exists b)X \vdash (\exists b)(\exists a)X$$
$$(\exists a)(\forall b)X \vdash (\forall b)(\exists a)X$$

are provable. Similarly for such permutations of quantifiers in any combination and for any number of quantifiers.

$$(\forall a)(\exists a)X \vdash (\exists b)(\forall a)X$$

is excluded since its intuitive interpretation is not generally valid.

The formulae in the axiomatic schemata 3 and 4 and in theorem *T4* solve the question on the substitution of $\forall$ and $\exists$ for each other. The axiomatic schemata 1, 2 and 7 and rule *R2* solve the problem of the introduction and elimination of quantifiers.

Another variant of S_c^1 is obtained if one takes instead of axiomatic schemata 2 and 6 the rule *R3*:

R3. If $X \vdash Y$ then $\sim Y \vdash \sim X$, where identical elementary K-formulae occur in X and Y.

5. OTHER SYSTEMS OF CLASSICAL THEORY OF QUANTIFIERS

The system S_c^2 of classical weakened theory of quantifiers is obtained through addition to S_c^1 of axioms which permit the obtaining of S^2 (or S_2) from S^1 (or S_1). K-formulae are involved in the restriction of the transitivity rule.

T1. All formulae provable in S_c^2 are generally valid.

T2. If $X \vdash Y$ is provable in S_c^2 then at least one identical elementary K-formula occurs in X and Y.

Systems S_c^3 and S_c^4 are obtained in a similar way. S_c^5 requires the following additions.

D1. $\vdash X$ is a formula of degenerate entailment if and only if X is a K-formula.

D2. The intuitive interpretation of $\vdash X$ is formed as follows:

1) if a freely occurs in X then $\vdash X$ is replaced by $(\forall a)X$; and so on for all subject and predicate variables;

2) the degenerate quantifiers are eliminated;

3) in the resulting formula $\vdash X'$ the formula X' is replaced by its complete interpretation.

D3. $\vdash X$ is generally valid if and only if X is a tautology with any number of designated variables for every variable occurring in X.

Additional inference rule:

R1. If $\vdash X$ then $(\forall a)X$,

where X is any K-formula and a is a subject or predicate variable.

D4. *R1* is observed in the definition of provable $\vdash X$.

T3. If $\vdash X$ is provable in S_c^5 then $\vdash X$ is generally valid.

In the case of S_c^6 all definitions for $X \rightarrowtail Y$ are analogous to definitions for $X \vdash Y$.

6. CLASSICAL THEORY OF QUANTIFIERS AND CLASSICAL PREDICATE CALCULUS

From the theorem of non-paradoxicality for S_c^i it follows that the formulae

$$(\forall s^1)\, P(s^1) \vdash P(s^2)$$
$$P(s^2) \vdash (\exists s^1)\, P(s^1)$$

are not provable in S_c^i. In general in these systems the formulae $X \vdash Y$, in which X and Y do not have identical occurrences of sentential variables and elementary subject-predicate formulae, are not provable. From this point of view our systems are narrower than the classical predicate calculus: not every formula $X \supset Y$, provable in the latter, corresponds to a formula $X \vdash Y$ provable in our systems.

It follows from *T3* of the previous section that the formulae

$$\vdash (\forall s^1)\, P(s^1) \supset P(s^2)$$
$$\vdash P(s^2) \supset (\exists s^1)\, P(s^1)$$
$$(\forall s^1)\, P(s^1) \rightarrowtail P(s^2)$$
$$P(s^2) \rightarrowtail (\exists s^1)\, P(s^1)$$

are not provable in S_c^i since their interpretations are not tautologies. Thus it is not the case that for every provable formula $X \supset Y$ in the classical predicate calculus one finds a provable formula $\vdash X \supset Y$ in S_c^5 and $X \rightarrowtail Y$ in S_c^6.

7. NON-CLASSICAL THEORY OF QUANTIFIERS

The systems S_n^i of non-classical theory of quantifiers are formed by the following additions to S_c^i.

Additions to the alphabet of the classical theory of quantifiers:

1) $\neg$ for intrinsic (partial) negation;

2) ? for indeterminacy;

3) { } for the possibility of constructing formulae.

D1. Addition to the definition of the K-formula: if X is a K-formula and a is a subject or predicate variable, then $(\neg\forall a)X$, $(\neg\exists a)X$, $(?\forall a)X$, $(?\exists a)X$ and $\{X\}$ are K-formulae.

D2. Proper formulae do not contain formulae of the type $\{X\}$.

D3. Intuitive interpretation:

1) $\{X1a\}\cdot\ldots\cdot\{Xna\}$ is the intuitive interpretation of $(\forall a)X$ with respect to a;

2) $\{X1a\} \vee \ldots \vee \{\sim Xna\}$ is the intuitive interpretation of $(\neg\forall a)X$ with respect to a;

3) $\sim(\{X1a\}\cdot\ldots\cdot\{Xna\}) \sim (\{\sim X1a\} \vee \ldots \vee \{\sim Xna\})$ is the intuitive interpretation of $(?\forall a)X$ with respect to a.

4) The intuitive interpretation of $(\exists a)X$, $(\neg\exists a)X$ and $(?\exists a)X$ is, respectively, the intuitive interpretation of $(\neg\forall a)\sim X$, $(\forall a)\sim X$ and $(?\forall a)\sim X$.

Semantic interpretation:

1) if one of $\{X\}$ and $\{\sim X\}$ has the value 1, then the other has the value 0;

2) if one of $\{X\}$ and $\{\sim X\}$ has the value 0, then the other can have the value 1 as well as the value 0 (i.e., both can have the value 0).

Axiomatic schemata replacing axiomatic schemata 3 and 4 of S_c^1:

1. $(\forall a)X \dashv\vdash (\neg\exists a)\sim X$
2. $(\neg\forall a)X \dashv\vdash (\exists a)\sim X$
3. $(?\forall a)X \dashv\vdash (?\exists a)\sim X$

Additional axiomatic schemata:

1. $\sim(\forall a)X \dashv\vdash (\neg\forall a)X \vee (?\forall a)X$
2. $\sim(\neg\forall a)X \dashv\vdash (\forall a)X \vee (?\forall a)X$
3. $\sim(?\forall a)X \dashv\vdash (\forall a)X \vee (\neg\forall a)X$

Considerations and theorems, analogous to those for S_c^1, hold for S_n^1.

8. INTUITIONIST LOGIC AND NON-CLASSICAL THEORY OF QUANTIFIERS

In intuitionist (constructivist) logic, as in classical logic, the formulae

$$p \supset (q \supset p), \quad \sim p \supset (p \supset q), \quad p \supset \sim\sim(q \vee \sim q),$$
$$\sim pp \supset q, \quad (\forall s^1)\, P(s^1) \supset P(s^2), \quad P(s^2) \supset (\exists s^1)\, P(s^1),$$

and others where the consequent contains variables lacking in the antecedent, are provable. This means that the interpretation of intuitionist implication as a sign of logical entailment generates paradoxes similar to those of material and strict implication. For this reason alone intuitionist logic cannot play the part of a general theory of deduction. We will not take up the possibility of using it as a special case of deductive theory.

Intuitionist logic has a more serious insufficiency relevant to its use as a general theory of logical entailment. The fact is that the idea of limiting the law of excluded middle (and the rule of double negation) arose on the basis of consideration of the internal structure of sentences (subjects, predicates, quantifiers) which is not taken into account in sentential logic. But the intuitionists did not introduce logical signs (of indeterminacy, of two types of negation) which make it possible to effect such a restriction in a natural way, not affecting sentential logic. They were therefore obliged to permit these limitations to sentential logic, as some sort of *a priori* premisses.

But the intuitionists' limitation of sentential logic is senseless because sentential logic has to provide exhaustive definitions of the logical signs "and", "not", "or", etc., and the above mentioned limitation means that the definitions of these signs remain partial and incomplete. As a result, intuitionist logic leaves not provable a class of formulae whose interpretation as rules of logical entailment leaves no room for doubt.

Thus, the interpretation of intuitionist logic as a theory of logical entailment produces a distorted (incorrect, "displaced") system: it accepts intuitively paradoxical rules and rejects intuitively indubitable rules of logical entailment.

The intuitionist ideas come out as natural consequences in our systems. The formulae

$$\sim(\neg\forall a)\, X \vdash (\forall a)\, X \quad \text{and} \quad \sim(\neg\exists a)\, X \vdash X$$

are not provable in them because their interpretation is not generally valid. As was shown above, the formula

$$\sim \urcorner P(s) \vdash P(s)$$

is not provable in the theory of predication.

9. WEAKENING OF INTUITIVE REQUIREMENTS

The requirement that in the case of strong implication the conclusion not contain elementary K-formulae lacking in the premisses can be weakened in what follows in the interests of simplicity of construction of the other sections of logic (logic of classes, modal logic, etc.). This can be done as follows: the conclusion should not contain variables lacking in the premisses. The same is the case for the other forms of logical entailment. In the contrary case it will be necessary to formulate the rules for modal sentences, sentences about classes, etc., not with the sign of logical entailment but with the sign of conditionality. But this does not influence the extension of the sets of such rules. This weakening can be presented as something having to do with the peculiarities of modal predicates, predicates of class-inclusion, predicates of order, etc., i.e., as a property of these constant logical predicates.

CHAPTER TWELVE

CONDITIONAL SENTENCES

1. CONDITIONAL SENTENCES

Conditional sentences are sentences of the type

"If X, then Y".

Their intuitive sense is the following: accepting (taking as true) X, the investigator also has to accept (take) Y. We will represent it (as above) with the symbol

$$X \rightarrow Y.$$

The sentence X is called the antecedent and Y the consequent.

D1. Negation and indeterminacy for such forms will be written by means of the symbols

$$\sim(X \rightarrow Y), \quad \urcorner(X \rightarrow Y), \quad ?(X \rightarrow Y).$$

D2. If X and Y are sentences, then $\alpha(X \rightarrow Y)$ is a sentence.

D3.
$$\begin{aligned} &\sim(X \rightarrow Y) \equiv \urcorner(X \rightarrow Y):?(X \rightarrow Y) \\ &\sim\urcorner(X \rightarrow Y) \equiv (X \rightarrow Y):?(X \rightarrow Y) \\ &\sim?(X \rightarrow Y) \equiv (X \rightarrow Y):(?(X \rightarrow Y)) \end{aligned}$$

The conditional sentence $\alpha(X \rightarrow Y)$ is made up of the sentences X and Y and the logical signs α and →. Here → is not a predicate (as distinct from $\vdash$).

2. THE CONSTRUCTION OF CONDITIONAL SENTENCES

Conditional sentences are obtained in the following ways:

1) as initial stipulations;
2) from logical implication;
3) from other sentences of the same type according to the rules of logic;

4) from sentences of another type according to the rules of logic;

5) from empirical investigation.

Point 1 needs no further explanation. Point 5 is explained in the Chapter on physical entailment. Point 4 is explained in the context of sentences of another type. Point 3 is cleared up in the following section.

Conditional sentences are formed from logical entailment as follows:

1) if $X \vdash Y$, then $X \rightarrow Y$

2) if $XZ \vdash Y$ and Z is true, then $X \rightarrow Y$.

$\vdash Z$ is not obligatory in the second point since there is not a $X \rightarrowtail Y$ corresponding to every $X \rightarrow Y$ (not to mention that $X \rightarrow Y$ can be stipulated, which is inadmissible in the case of $X \rightarrowtail Y$).

If $X \rightarrow Y$ is accepted by stipulation or obtained from accepted sentences according to the rules of a system with degenerate entailment, we will use symbols like

$$\vdash (X \rightarrow Y).$$

3. TRUTH-VALUES

Solution of the question on the truth-values of $\alpha(X \rightarrow Y)$ depends on the means of the construction thereof. Thus, if it is obtained by stipulation it is considered true; if $XZ \vdash Y$ and Z is true, then $X \rightarrow Y$ is true, etc. In some cases the truth-values of $\alpha(X \rightarrow Y)$ can be established relative to those of X and Y. Thus, if X is true and Y is false, then $X \rightarrow Y$ is false. But $X \rightarrow Y$ is not a truth-function of X and Y. $X \rightarrow Y$ can be as well true as false when X and Y are both true or both false. $X \rightarrow Y$ is not identical with $X \supset Y$: $X \rightarrow Y$ can be false with a false X and can be false with a true Y. If both X and Y are true this in itself does not make $X \rightarrow Y$ true. To what extent the repetition of such a situation makes it possible to accept conditional sentences will be taken up below.

4. LOGICAL CONDITIONS

D1. $\downarrow X$ is an active condition of $\downarrow Y$, if and only if $X \rightarrow Y$.

D2. $\downarrow X$ is a passive condition of $\downarrow Y$, if and only if $\sim X \rightarrow \sim Y$.

D3. $\downarrow X$ is a full condition of $\downarrow Y$, if and only if $(X \rightarrow Y) \cdot (\sim X \rightarrow \sim Y)$.

D4. $\downarrow X^1, \ldots, \downarrow X^n$ are sufficient conditions of $\downarrow Y$ $(n \geqslant 1)$, if and only if $X^1 \cdot \ldots \cdot X^n \rightarrow Y$.

D5. $\downarrow X^1,\ldots,\downarrow X^n$ are necessary conditions of $\downarrow Y$, if and only if $\sim(X_1\cdot\ldots\cdot X^{n-1}\rightarrow Y)$, where $X_1,\ldots,X_{n-1}$ are any $n-1$ sentences from $X^1,\ldots,X^n$.

5. DEDUCTIVE PROPERTIES OF CONDITIONAL SENTENCES

The system S_{if} is formed by the following additions to our earlier systems:

Alphabet:

1) $\rightarrow$ is the sign of conditionality (the sign of "if ..., then ...");

2) $\urcorner$ for intrinsic (partial) negation;

3) ? for indeterminacy.

D1. $X\rightarrow Y$, $X\rightarrow\sim Y$, $\sim X\rightarrow Y$ and $\sim X\rightarrow\sim Y$ are elementary conditional formulae if and only if X and Y are sentential variables.

D2. Conditional formula:

1) if X is a sentential formula and Y is formed from X by substitution of a conditional formula for at least one occurrence of a sentential variable, then Y is a conditional formula;

2) if at least one of X and Y is a conditional formula then $X\rightarrow Y$ is a conditional formula;

3) if $X\rightarrow Y$ is a conditional formula, then $\urcorner(X\rightarrow Y)$ and $?(X\rightarrow Y)$ are conditional formulae.

Axioms of S_{if}:

1. $(p\rightarrow q)\,p\vdash q$
2. $(p\rightarrow q)\vdash(\sim q\rightarrow\sim p)$
3. $(p\rightarrow q)(q\rightarrow r)\vdash(p\rightarrow r)$
4. $(pq\rightarrow r)\dashv\vdash(p\rightarrow(q\rightarrow r))$
5. $(p\rightarrow qr)\dashv\vdash(p\rightarrow q)(p\rightarrow r)$
6. $\sim(p\rightarrow q)\dashv\vdash\urcorner(p\rightarrow q)\vee ?(p\rightarrow q)$
7. $\sim\urcorner(p\rightarrow q)\dashv\vdash(p\rightarrow q)\vee ?(p\rightarrow q)$
8. $\sim ?(p\rightarrow q)\dashv\vdash(p\rightarrow q)\vee\urcorner(p\rightarrow q)$

Rules of inference of S_{if}:

R1. Substitution of conditional formulae for sentential variables.

R2. If $X\vdash Y$, then $\vdash(X\rightarrow Y)$.

D3. A conditional formula is provable in S_{if} only in function of the axioms and rules of inference of S_{if}.

The classical case is obtained by rejecting axioms 11–17 or by accepting the additional axioms, $\vdash {\sim} ?(p \rightarrow q)$ and ${\sim}(p \rightarrow q) \vdash \neg(p \rightarrow q)$.

Another form of writing S_{if} without the signs $\neg$ and ? is obtained as follows. Instead of point 3 in *D2*, we take the following: if $X \rightarrow Y$ is a conditional formula, then $X {\sim}{\rightarrow} Y$ is a conditional formula. In such a case the symbol $X {\sim}{\rightarrow} Y$ is equivalent to $\neg(X \rightarrow Y)$ and the symbol ${\sim}(X \rightarrow Y) {\sim} \neg(X \rightarrow Y)$ is equivalent to $?(X \rightarrow Y)$. Such a notation makes clear the difference between the classical and non-classical cases: in the classical case the third possibility, ${\sim}(X \rightarrow Y) {\sim} (X {\sim}{\rightarrow} Y)$, is excluded, i.e., one accepts ${\sim}(X \rightarrow Y) \dashv\vdash (X {\sim}{\rightarrow} Y)$ and $\vdash {\sim}({\sim}(X \rightarrow Y)\ {\sim}(X {\sim}{\rightarrow} Y))$.

Interpretation of conditional formulae:

1) If $X \rightarrow Y$ takes the value v^1 and X takes the value v^1 (Y takes nv^1), then Y has to take the value v^1 (X, the value nv^1);

2) If $X \rightarrow Y$ takes nv^1 and X takes v^1 (Y takes nv^1), then one can assign to Y either v^1 or nv^1 (for X, either nv^1 or v^1), and both possibilities have to be taken into consideration;

3) If we assign to X the value v^1 (to Y the value nv^1) and we are obliged (because of the presuppositions holding for the explanation of the values of formulae occurring in $X \rightarrow Y$) to assign to Y the value v^1 (to X the value nv^1), then $X \rightarrow Y$ has to have the value v^1;

4) If we assign to X the value v^1 (to Y the value nv^1) and there is no need of assigning the value v^1 to Y(nv^1 to X), i.e., we can assign nv^1 to Y (v^1 to X), then it is necessary to assign nv^1 to $X \rightarrow Y$.

5) $?(X \rightarrow Y)$ is equivalent to ${\sim}(X \rightarrow Y) {\sim} \neg(X \rightarrow Y)$; if one of $(X \rightarrow Y)$ and $\neg(X \rightarrow Y)$ has the value v^1, the other has the value nv^1; but, if one of them has the value nv^1, then the value of the other is indeterminate.

6) Value is assigned to a conditional formula in the same way as to a variable (keeping point v^1 in mind); here two elementary conditional formulae $X^1 \rightarrow X^2$ and $Y^1 \rightarrow Y^2$ are distinct if and only if X^1 and Y^1 are different or X^2 and Y^2 are different (or both).

T1. All formulae provable in S_{if} are tautologies.

T2. The system S_{if} is non-paradoxical in the same sense as S^1–S^4 and S_1–S_4.

The problem of completeness of the system S_{if} has not been investigated.

6. CONTRAFACTUAL SENTENCES

Sentences of the type "If X were not, Y would not be" are shorthand for

$$XY(\sim X \rightarrow \sim Y)$$

7. EXPLANATION

To explain $\downarrow Y$ means to find a set of acceptable sentences X such that

$$X \rightarrow Y.$$

8. CONDITIONALITY AND QUANTIFIERS

There is a connection between quantifiers and the sign of conditionality. In particular,

1. $(X \rightarrow Y) \rightarrow (\forall \downarrow X)(X \rightarrow Y)$
2. $(\forall s \downarrow P^1)\, P^2(s \downarrow P^1) \leftrightarrow (P^1(s) \rightarrow P^2(s))$
3. $(\forall s)\, P(s) \leftrightarrow ((s \rightharpoonup s^i) \rightarrow P(s^i))$

CHAPTER THIRTEEN

THEORY OF TERMS

Alphabet:

1) $s, s^1, s^2, \ldots$ are subject variables;

2) $P, P^1, P^2, \ldots$ are predicate variables;

3) $t, t^1, t^2, \ldots$ are term variables;

4) *sc* is a subject constant ("object");

5) *Pc* is a predicate constant ("attribute");

6) $\rightharpoonup$ is a two-place predicate constant ("... is included as to value in ...").

D1. Subjects:

1) subject variables and *sc* are subjects;

2) if $a^1, \ldots, a^n$ are subjects, then $(\sim a^1)$, $(a^1 \cdot \ldots \cdot a^n)$, $(a^1 \vee \ldots \vee a^n)$, $(\cdot a^1, \ldots, a^n)$, $(\vee a^1, \ldots, a^n)$ and $(a^1, \ldots, a^n)$ are subjects;

3) if X is a formula and a is a subject, then SX and $a \downarrow X$ are subjects;

4) if a is a term, then $[a]$ is a subject.

D2. Predicates:

1) predicate variables and *Pc* are predicates;

2) if $a^1, \ldots, a^n$ are predicates, then $(\sim a^1)$, $(a^1 \cdot \ldots \cdot a^n)$, $(a^1 \vee \ldots \vee a^n)$, $(\cdot a^1, \ldots, a^n)$, $(\vee a^1, \ldots, a^n)$, $(\tilde{a}^1)$ are predicates;

3) if X is a formula and a is a predicate then PX and $a \downarrow X$ are predicates;

4) $\rightharpoonup$ is a predicate.

D3. Terms:

1) term variables are terms;

2) subjects and predicates are terms;

3) if $a^1, \ldots, a^n$ are terms, then $(\sim a^1)$, $(a^1 \cdot \ldots \cdot a^n)$, $(a^1 \vee \ldots \vee a^n)$, $(\tilde{a}^1)$, $(\cdot a^1, \ldots, a^n)$ and $(\vee a^1, \ldots, a^n)$ are terms;

4) if a is a term, then $[a]$ is a term.

D4. $Q(a)$ is an elementary K-formula if and only if Q is a predicate and a is a subject.

D5. $\rightharpoonup([a], [b])$ is an elementary formula of the theory of terms if and only if a and b are terms. In what follows we will use the more graphic symbols $[a] \rightharpoonup [b]$.

D6. $[a] \rightleftharpoons [b]$ is shorthand for $([a] \rightharpoonup [b])\,([b] \rightharpoonup [a])$.

D7. $a{:}b$ is short for $a \cdot \sim b \vee \sim a \cdot b$; $a^1{:}a^2{:}\ldots{:}a^n$ is short for $a^1 \cdot \sim a^2 \cdot \ldots \cdot \sim a^n \vee a^2 \cdot \sim a^1{:}\ldots{:}\sim a^n \vee \ldots \vee a^n \cdot \sim a^1 \cdot \ldots \cdot \sim a^{n-1}$.

In what follows the square brackets will be dropped, on the assumption that in formulae with the predicate $\rightharpoonup$ they can always be restored. The sign $\cdot$ will also be dropped as in the general theory of deduction.

Axioms *AI*:

1. $\vdash \sim \sim t \rightleftharpoons t$
2. $\vdash t^1 \rightharpoonup t^1 t^2$
3. $\vdash t^1 t^2 \rightharpoonup t^2 t^1$
4. $\vdash t^1 \ldots t^i t^k \ldots t^n \rightleftharpoons t^1 \ldots t^i (t^k \ldots t^n)$ where $i \geqslant 1, n \geqslant 3$,
5. $\vdash \sim (t^1 t^2) \rightleftharpoons \sim t^1 \vee \sim t^2$
6. $\vdash (t^1 \rightharpoonup t^2) \to (\sim t^2 \rightharpoonup \sim t^1)$
7. $\vdash (t^1 \rightharpoonup t^2)(t^2 \rightharpoonup t^3) \to (t^1 \rightharpoonup t^3)$
8. $\vdash (t^1 \rightharpoonup t^3)(t^2 \rightharpoonup t^3) \to (t^1 t^2 \rightharpoonup t^3)$
9. $\vdash (t^1 \vee \sim t^1) \rightharpoonup t^2$

Axioms *AII*:

1. $\vdash \alpha(\cdot P^1, \ldots, P^n)(s) \leftrightarrow \alpha P^1(s) \cdot \ldots \cdot \alpha P^n(s)$
2. $\vdash \alpha(\vee P^1, \ldots, P^n)(s) \leftrightarrow \alpha P^1(s) \vee \ldots \vee \alpha P^n(s)$
3. $\vdash \alpha P(\cdot s^1, \ldots, s^n) \leftrightarrow \alpha P(s^1) \cdot \ldots \cdot \alpha P(s^n)$
4. $\vdash \alpha P(\vee s^1, \ldots, s^n) \leftrightarrow \alpha P(s^1) \vee \ldots \vee \alpha P(s^n)$
5. $\vdash \alpha(\widetilde{P})(s) \leftrightarrow \sim \alpha P(s)$
6. $\vdash (P^1 \cdot P^2)(s) \to P^1(s) \cdot P^2(s)$
7. $\vdash P(s^1 \cdot s^2) \to P(s^1) \cdot P(s^2)$
8. $\vdash (P^1 \vee P^2)(s) \leftrightarrow P^1(s) \vee P^2(s)$
9. $\vdash P(s^1 \vee s^2) \leftrightarrow P(s^1) \vee P(s^2)$
10. $\vdash \neg P^1(s) \vee \neg P^2(s) \to \neg (P^1 \cdot P^2)(s)$
11. $\vdash \neg P(s^1) \vee \neg P(s^2) \to \neg P(s^1 \cdot s^2)$
12. $\vdash \neg (P^1 \vee P^2)(s) \leftrightarrow \neg P^1(s) \neg P^2(s)$
13. $\vdash \neg P(s^1 \vee s^2) \leftrightarrow \neg P(s^1) \neg P(s^2)$
14. $\vdash P(s^1) \cdot ?P(s^2) \to ?P(s^1 \cdot s^2)$
15. $\vdash P^1(s) \cdot ?P^2(s) \to ?(P^1 \cdot P^2)(s)$
16. $\vdash ?P(s^1) \cdot ?P(s^2) \to ?P(s^1 \cdot s^2)$
17. $\vdash ?P^1(s) \cdot ?P^2(s) \to ?(P^1 \cdot P^2)(s)$.

For the sake of brevity in the notation of the following axioms we will use the symbols α, β and γ in the following sense: 1) each of these symbols by itself will indicate the presence or absence of the signs $\sim$, $\neg$ or ?; 2) if they are met together in the same formula then they indicate any of the possible combinations of the signs $\neg$ and ? and their absence, if it is not stated that they are different.

D8. $s\alpha\downarrow P$ is short for $s\downarrow(\alpha P(s))$
$P\alpha\downarrow s$ is short for $P\downarrow(\alpha P(s))$

Axiomatic schemata *AIII*:

1. $\vdash (t\downarrow X)\downarrow Y \rightleftharpoons (t\downarrow Y)\downarrow X$
2. $\vdash t \rightharpoonup t\downarrow X$,
3. $\vdash t\downarrow X \rightharpoonup t$, where t does not occur freely in X and where X and Y are any formulae.

Axioms *AIV*:

1. $\vdash (s^1 \rightharpoonup s^2)(\forall s^1)\,\alpha P(s^1) \rightarrow (\forall s^2)\,\alpha P(s^2)$
2. $\vdash (s^1 \rightharpoonup s^2)(\exists s^2)\,\alpha P(s^2) \rightarrow (\exists s^1)\,\alpha P(s^1)$
3. $\vdash (P^1 \rightharpoonup P^2)(\forall s)\,P^2(s) \rightarrow (\forall s)\,P^1(s)$
4. $\vdash (P^1 \rightharpoonup P^2)(\exists s)\,P^2(s) \rightarrow (\exists s)\,P^1(s)$
5. $\vdash (P^1 \rightharpoonup P^2)\neg P^1(s) \rightarrow \neg P^2(s)$
6. $\vdash (P^1 \rightharpoonup P^2)?\,P^1(s) \rightarrow \sim P^2(s)$
7. $\vdash (\alpha P^1(s) \rightarrow \beta P^2(s)) \rightarrow \beta P^2(s\alpha\downarrow P^1)$
8. $\vdash (\alpha P(s^1) \rightarrow \beta P(s^2)) \rightarrow \beta(P\alpha\downarrow s^1)(s^2)$

Axioms *AV*:

1. $\vdash \sim s^1 \rightharpoonup (s^2\neg\downarrow(P\downarrow s^1))$
   $\vdash \sim s^1 \rightharpoonup (s^2?\downarrow(P\downarrow s^1))$
2. $\vdash \sim s^1 \rightharpoonup (s^2\downarrow(P\neg\downarrow s^1))$
   $\vdash \sim s^1 \rightharpoonup (s^2\downarrow(P?\downarrow s^1))$
3. $\vdash \alpha P(s^1)\,\beta P(s^2) \rightarrow \sim (s^1 \rightharpoonup s^2)$,

where α and β are different.

4. $\vdash \alpha P^1(s)\,\beta P^2(s) \rightarrow \sim (P^1 \rightharpoonup P^2)$

Axioms *AVI*:

1. $\vdash Pc \rightharpoonup P$
2. $\vdash sc \rightharpoonup s$

3. $\vdash sc\downarrow(Pc\downarrow s)\rightharpoonup s$
4. $\vdash Pc\downarrow(sc\downarrow P)\rightharpoonup P$
5. $\vdash \alpha P(s)\rightarrow(sc\alpha\downarrow P\rightharpoonup s)$
6. $\vdash \alpha P(s)\rightarrow(Pc\alpha\downarrow s\rightharpoonup P)$

Axioms *AVII*:

1. $\vdash(s^1, s^2, \ldots, s^n)\rightleftharpoons(s^1, (s^2, \ldots, s^n))$
2. $\vdash(s^1, \ldots, s^{n-1}, s^n)\rightleftharpoons((s^1, \ldots, s^{n-1}), s^n)$
3. $\vdash(s^1, \ldots, s^i, \ldots, s^k, \ldots, s^n)\rightleftharpoons(s^1, \ldots, (s^i, \ldots, s^k), \ldots, s^n)$

Axioms *AVIII*:

1. $\vdash \alpha(P\alpha\downarrow s)\,(s)$
2. $\vdash \alpha P(s\alpha\downarrow)P$

Axioms *AIX*:

1. $\vdash(\forall s)\, P^1(s\downarrow P^2)\leftrightarrow(\forall s\downarrow P^2)\, P^1(s\downarrow P^2)$
2. $\vdash(\exists s)\, P^1(s\downarrow P^2)\leftrightarrow(\exists s\downarrow P^2)\, P^1(s\downarrow P^2)$
3. $\vdash P(s^1, \ldots, s^n)\leftrightarrow P(P(s^1, \ldots, s^n))\,(s^i) \quad (i = 1, \ldots, n)$
   $\vdash P(s^1, \ldots, s^n)\leftrightarrow P(P(s^1, \ldots, s^n))\,(s^i, \ldots, s^k) \quad (i = 1, \ldots, n;$
   $k = 1, \ldots, n)$

R1. Substitution of terms for term variables.

R2. Substitution of $\rightharpoonup$ for two-place predicate variables.

R3. If $t^1\rightharpoonup t^2$, and b is formed from a by putting t^2 for t^1, where t^1 occurs in a, then $a\rightharpoonup b$.

R4. If $t^1\rightleftharpoons t^2$, and Y is formed from X by putting t^2 for t^1, then $\vdash(X\rightarrow Y)$.

R5. If $\vdash(X\leftrightarrow Y)$, then $SX\rightleftharpoons SY$ and $PX\rightleftharpoons PY$; and vice versa.

We have not investigated the formal properties of the theory of terms. Our exposition is only an indication of the direction in which a complete theory of terms could be developed.

CHAPTER FOURTEEN

CLASSES

1. CLASSES

D1. If t is an individual term, then the object Π corresponding to it is an individual.

D2. If t^i is an individual term from the value-range of t, and Π^i is the object it designates, then Π^i is an individual from the value-range of t.

D3. To form (and select) a class of individuals means to construct a term "The class of individuals from the value-range of t", where t is the term in question. We will write such a term with the symbol

$$Kt,$$

where K is the class-generative sign. Individuals from the value-range of t are elements of Kt. The words "class" and "set" are synonymous.

A1. Kt is a term of a class if and only if t is a term.

A2. Kt is an individual term.

T1. A class of individuals is an individual.

To form a class of individuals means literally to say "The class of individuals from the value-range of such and such a term (such terms)", i.e., to construct an appropriate term. Different languages can do this differently but the heart of the matter remains the same. In order that a class exist it is sufficient to form it. For example, in constructing the expression "the class of gods" we form the class of gods and this class exists as a special object even though we know that gods do not empirically exist.

The difference between class and n-tuple is clear from such an example. A triple of integers such that the sum of the cubes of two of them is equal to the cube of the third does not exist. But the class of such triples exists as long as we construct the expression "The class of triples of integers such that the sum of the cubes of two of them is equal to the cube of the third". This class can be studied as a special, existing object.

The following principles form part of the definition of class and have to be taken into consideration in class-formation:

A3. Every element of a class can be selected independently of the formation (and selection) of the class itself (principle of the independence of the elements from the class);

A4. Relative to any individual it is possible to establish whether or not it is an element of a given class (principle of determinacy).

D4. A class of individuals exists (does not exist) if it is formed (not formed) in accord with *D3* and the principles of independence and determinacy.

As a result of the principle of the independence of the element from the class, a class cannot be an element of itself. Whence we conclude that there is no class of all classes. The class of all classes which are their own element also does not exist. This principle eliminates the famous paradoxes of set-theory.

If one can find even one individual, about which one cannot say whether or not it is an element of some class, then the class in question does not exist (is incorrectly formed).

D5. A class is empirical if and only if all its elements are empirical objects and it is abstract if and only if all its elements are abstract objects.

The formation of classes from empirical and abstract objects (mixed classes) is not excluded.

In the formation of empirical classes one indicates the time and space in which its elements exist. If such an indication is lacking, then any time and space are assumed.

A5. $\sim(Kt \leftarrow E) \equiv (Kt \urcorner\leftarrow E)$

T2. $(Kt \leftarrow E){:}(Kt \urcorner\leftarrow E)$.

The sentence that an individual of the value-range of t^1 is an element of Kt^2 (is included in Kt^2) will be written as

$$t^1 \in Kt^2$$

and its negation as

$$t^1 \urcorner \in Kt^2.$$

The sentence $t^1 \in Kt^2$ (and its negation) is a sentence with two subjects, t^1 and Kt^2, and a two-place predicate (similarly for the negation), "The first is included in the second" or "The first is an element of the second". And t^1 is not necessarily an individual term. Hence, these sentences can be local in the case of empirical objects.

A6. $\sim(t^1 \in Kt^2) \equiv t^1 \neg \in Kt^2$
T3. $(t^1 \in Kt^2) : (t^1 \neg \in Kt^2)$
A7. $(t^1 \in K \sim t^2) \rightarrow \sim(t^1 \in Kt^2)$
A8. $\sim(\sim t^3 t^3 \rightharpoonup t^1) \rightarrow (\sim(t^1 \in Kt^2) \rightarrow (t^1 \in K \sim t^2))$

The symbol *K* (and the word "class") in the above sense is not a term of that science in which *t* and *Kt* are terms: it is a logical tool for the formation of a new term from a given term *t* and thereby a tool for the isolation (selection) of a special object, i.e., of all individuals of the value-range of *t*.

2. INCLUSION IN A CLASS

Inclusion in a class is carried out by agreement and in accord with the rules of logic. The following are normally used:

A1. $(t \rightharpoonup t^i) \rightarrow (t^i \in Kt)$
A2. $(s \leftarrow PX^0) \leftrightarrow (s \in Ks^* \downarrow PX^0(s/s^*))$.

3. CLASSES OF CLASSES

A class can have some classes as its elements. It is important to keep the following in mind. If we form a class of classes then the elements of this class are all classes which are formed according to the above rules and can be selected independently of the class in which they are included as elements. And such a class cannot include all classes according to the very concept of "class".

To form a class of classes one must carry out one of two operations:

1) form the term "The class, the elements of which are classes $Kt^1, \ldots, Kt^n$",

where $n \geqslant 2$ and all the terms $Kt^1, \ldots, Kt^n$ are terms of classes;

2) form a term t^m such that

$$t^m \rightharpoonup Kt^1, \ldots, t^m \rightharpoonup Kt^n,$$

and then construct a term

$$Kt^m.$$

Examples of t^m: "empty class", "finite class", "infinite class", etc. We obtain a corresponding Kt^m: "class of empty classes", "class of finite classes", etc.

The fact that a term is a term of a class of classes often remains hidden in ordinary and scientific language. Let us take, for example, the expressions "The class of officers serving in one regiment", "The class of members of one party", "The class of molecules in a given region of space", etc. Though they seem to have the structure Kt where t is not the term of a class, they really are terms of the type t^m. Thus, the term "The class of officers serving in one regiment" is generic in comparison to "The class of officers of the 110th regiment", "The class of officers of the 109th regiment", etc., and is not a term of class of the type Kt.

The term "class" (we use k for it) intuitively means the following: if a is a term, then Ka is k, i.e.,

$$k \rightharpoonup Ka\,.$$

Whence one obtains that if a is a term, then

$$Ka \in Kk$$

i.e., every class is an element of the class of classes. And

$$Kk \in Kk$$

is clear. However, this reasoning contains errors.

First of all, one has to distinguish the term "class" (k) from the class-generative operator "class" (K) which is not a term. The definition of the term k has the following form:

D1. Let k be a term such that if a is a term, then

$$k \rightharpoonup Ka\,.$$

Since

$$(k \rightharpoonup Ka) \rightarrow (Ka \in Kk)\,,$$

the definition can take the following form:

D′1. Let k and Kk be terms such that if a is a term, then

$$Ka \in Kk\,.$$

The expression "Let k be a term" has definite logical properties: it converts things of the type k, which previous to this and independently of it were not terms, into terms. And the expression "if a is a term" makes it possible to take as a only such things which are terms or which become terms independently of *D1*. In short, *D1* is a definition with a variable, the rule for which was indicated in the third chapter. Here a plays the role of a variable (its value-range consists of terms which do not depend on k).

According to the rule of construction of definitions of this type from *D1* one cannot obtain the consequence "If a is a term then $k \rightharpoonup Ka$ (or $Ka \in Kk$)" or $(\forall a)(k \rightharpoonup Ka)$ and $(\forall a)(Ka \in Kk)$, where a is any term, including the term k. Nor can one obtain the assertions $k \rightharpoonup Kk$ and $Kk \in Kk$. From *D1* one can obtain only the theorem:

T1. If a is a term which does not depend in value on k (i.e., the value of which can be established without *D1*), then $k \rightharpoonup Ka$ (then $Ka \in Kk$).

The question on the status of the above assertions depends on circumstances extrinsic to them: they can be accepted or not as axioms depending on whether or not they are needed and whether or not they lead to contradiction.

4. PARADOX OF THE CLASS OF NORMAL CLASSES

The expression "normal class" (or "normal set") is defined as follows: a class is called normal if and only if it is not an element of itself. This definition is unsuitable because it does not clearly express that a class is always a class of something. We adopt the following definition to get rid of this insufficiency:

D1. If a is a term and $\sim(Ka \in Ka)$, then Ka will be called a normal class (instead of the expression "normal class" we will use n).

The expression "will be called" has logical properties which are clearly expressed in the case of *D1* as follows:

D′1. Let n be a term such that $n \rightharpoonup Ka$ (or $Ka \in Kn$), if and only if a is a term and $\sim(Ka \in Ka)$.

Again, we have here a definition with variables: a here plays the role of variable; the value-range of a are terms which do not depend in value on n.

In obtaining the paradox of the class of normal classes one forgets (or does not notice) that in place of a one cannot put the term n and any other term defined through n. If this is forgotten, the definition assumes the form of assertion A: if a is a term and $\sim(Ka \in Ka)$, then $Ka \in Kn$; if a is a term and $Ka \in Ka$, then $\sim(Ka \in Kn)$. Putting the term n for a, we obtain the assertion B: if $\sim(Kn \in Kn)$, then $Kn \in Kn$; if $Kn \in Kn$, then $\sim(Kn \in Kn)$.

But the assertion A is not true. The following consequence of *D1* is true: if a is a term, independent in value from n, and $\sim(Ka \in Ka)$, then $Ka \in Kn$; if $Ka \in Ka$, then under the same conditions relative to a, $\sim(Ka \in Kn)$. But since n depends in value on itself (we cannot know the value of n without defining n), it is not possible to obtain B.

The question as to whether or not to accept the assertion $n \rightharpoonup Kn$ (and its consequence, $Kn \in Kn$) remains open. It is irrelevant to *D1* in the sense that having accepted $n \rightharpoonup Kn$ and *D1* we still cannot obtain a logical contradiction. Neither do we obtain a contradiction if we take *D1* and $\sim(Kn \in Kn)$ (and its consequence, $\sim(n \rightharpoonup Kn)$).

5. LIMITATIONS OF THE CONCEPT OF CLASS

Once Kt is formed, there is no difficulty in also constructing t^i which will be considered individual and for which $t^i \in Kt$. The question on the existence of the individuals designated by that term remains open. Hence, there could be non-existent individuals among the elements of a class. This is of some practical inconvenience. In practice one operates with a much more narrow concept of class: the class includes the existing individuals of the value-range of a given term. We will call such a class existential and define it as:

D1. $K^e t = Df. K(t \downarrow E)$.

Further we will say that still another notion of class is possible, intermediate between Kt and $K^e t$; this is the notion of potential class which we represent as $K^M t$ and define as

D2. $K^M t = Df. K(t \downarrow M)$,

where $t \downarrow M$ reads "t which is possible" (see below for the notion of possibility). Potential classes can in turn be divided according to type of possibility into logical or factual ("physical").

6. EMPTY AND NON-EMPTY CLASSES

D1. $K^e t$ will be considered empty if for any t^i such that $t \rightharpoondown t^i$ $\sim (t^i \leftarrow E)$ is the case, and non-empty if there is a t^i such that $(t \rightharpoondown t^i) \cdot (t^i \leftarrow E)$. Existence and non-existence are here used as they were in the definition of a class (i.e., not necessarily existence in the present, possibly existence in the past, not necessarily empirical existence, etc.).

D2. A definition of the emptiness or non-emptiness of $K^M t$ is obtained from *D1* by putting M for E.

D3. Kt is existentially (potentially) empty if and only if $K^e t (K^M t)$ is empty.

D4. Kt is empty if and only if there is no individual term t^i such that

$$(t^i \in Kt).$$

T1. If Kt is empty then it is existentially and potentially empty.

For example, the class of high-jumpers making the 2 meter 30 centimeter mark is existentially empty. If this class is existential, then it is simply empty. If, however, it is potential, then it is not empty. If $K^e t$ is empty then there is no element in it (it is impossible to name or in any way select an individual t^i such that $t^i \in K^e t)$ since $(t \downarrow E) \leftarrow E$. But a case where it would be impossible to name an individual included in Kt seems doubtful: the investigator can always arbitrarily construct a term t^i such that it is individual (in accord with the accepted norms) and for which $t \rightharpoondown t^i$ is valid; if Kt is existentially empty, then $\sim (t^i \leftarrow E)$. So that the definition of an empty class as a class which has no elements needs some extra-logical limitations: what is said in the definiens of *D4* is valid by virtue of agreements made in a given science and of the consequences flowing therefrom.

7. UNIVERSAL CLASSES

The definition of the universal class as the class in which are included all individuals leads to contradictions since it is impossible (in view of the above agreements) to construct such a class. Therefore, we have to assume that such a class is formed in accord with the accepted rules, i.e., as already given.

D1. Kt is universal if and only if $t^i \in Kt$ for any individual t^i.

The universal class is not a class of all individuals. Its definition has the following sense: if we construct a term Kt and if for any individual for which the term t^i is constructed $t^i \in Kt$ holds, then this Kt is considered universal. This assumes that one observes the rules for the construction of terms (and for the formation of classes).

By analogy with the definitions of existentially and potentially empty classes:

D2. $K^e t(K^M t)$ is universal if and only if for any individual t^i

$$(t^i \downarrow E) \in K^e t \quad ((t^i \downarrow M) \in K^M t).$$

D3. Kt is existentially (potentially) universal if and only if $K^e t(K^M t)$ is universal.

T1. If Kt is universal, then it is existentially and potentially universal.

8. DERIVATIVE CLASSES

D1. Classes which are designated by terms like Kt are primitive classes. And the terms of Kt are primitive terms of classes (or terms of primitive classes).

With the help of primitive terms of classes, one forms derivative terms of classes (terms of derivative classes; derivative classes). This is usually seen as follows:

1) $Kt^1 \cap Kt^2$ is a logical product of Kt^1 and Kt^2; similarly $Kt^1 \cap Kt^2 \cap \ldots \cap Kt^n$ is a logical product of $Kt^1, \ldots, Kt^n$ $(n \geqslant 2)$;

2) $Kt^1 \cup Kt^2$ is the logical sum of Kt^1 and Kt^2; similarly $Kt^1 \cup Kt^2 \cup \ldots \cup Kt^n$ is the logical sum of $Kt^1, \ldots, Kt^n$.

3) $\overline{K}t$ is the logical complement of Kt

D2. $Kt^1 \cap Kt^2$ is a class such that for any individual t^i

$$(t^i \in Kt^1) \cdot (t^i \in Kt^2) \leftrightarrow (t^i \in Kt^1 \cap Kt^2);$$

similarly, $Kt^1 \cap Kt^2 \cap \ldots \cap Kt^n$ is a class such that for any individual t^i

$$(t^i \in Kt^1) \cdot (t^i \in Kt^2) \cdot \ldots \cdot (t^i \in Kt^n) \leftrightarrow (t^i \in Kt^1 \cap Kt^2 \cap \ldots \cap Kt^n).$$

D3. $Kt^1 \cup Kt^2$ is a class such that for any individual t^i

$$(t^i \in Kt^1) \vee (t^i \in Kt^2) \leftrightarrow (t^i \in Kt^1 \cup Kt^2);$$

similarly $Kt^1 \cup Kt^2 \cup \ldots \cup Kt^n$ is a class such that for any individual t^i

$$(t^i \in Kt^1) \vee (t^i \in Kt^2) \vee \ldots \vee (t^i \in Kt^n) \leftrightarrow$$
$$\leftrightarrow (t^i \in Kt^1 \cup Kt^2 \cup \ldots \cup Kt^n).$$

D4. $\overline{Kt}$ is a class such that for any individual t^i

$$(t^i \in Kt):(t^i \in \overline{Kt}).$$

D5. $Kt^1 \cup \ldots \cup Kt^n$, $Kt^1 \cap \ldots \cap Kt^n$, $\overline{Kt}$ are terms of classes. Formation of them is formation of classes with the properties indicated in *D2*, *D3* and *D4*.

The term "The class of individuals from the value-range of $t^1, \ldots, t^n$" (or "The class containing $t^1, \ldots, t^n$") is a special case of $Kt^1 \cup \ldots \cup Kt^n$.

A1. $\overline{Kt} \rightleftharpoons K \sim t$
A2. $Kt^1 \cap Kt^2 \rightleftharpoons K(t^1 t^2)$
$Kt^1 \cap Kt^2 \cap \ldots \cap Kt^n \rightleftharpoons K(t^1 t^2 \ldots t^n)$.
A3. $Kt^1 \cup Kt^2 \rightleftharpoons K(t^1 \vee t^2)$
$Kt^1 \cup Kt^2 \cup \ldots \cup Kt^n \rightleftharpoons K(t^1 \vee t^2 \vee \ldots \vee t^n)$

9. RELATIONS BETWEEN CLASSES

Relations between classes are not the same as those between terms although there is some connection. Thus, $Kt^1 \rightharpoonup Kt^2$ does not follow from $t^1 \rightharpoonup t^2$: the former are individual terms; and if $Kt^1 \rightharpoonup Kt^2$ then $Kt^2 \rightharpoonup Kt^1$ – which is not always true for $t^1 \rightharpoonup t^2$.

We assume the designations:

1) $Kt^1 \subset Kt^2$ for: Kt^1 is included in Kt^2;
2) $Kt^1 \subset\supset Kt^2$ for: Kt^1 and Kt^2 coincide.

D1. $Kt^1 \subset Kt^2$ if and only if for any individual t^i

$$(t^i \in Kt^1) \rightarrow (t^i \in Kt^2).$$

D2. $Kt^1 \subset\supset Kt^2 \equiv (Kt^1 \subset Kt^2)\cdot(Kt^2 \subset Kt^1)$
A1. $(t^1 \rightharpoonup t^2) \leftrightarrow (Kt^2 \subset Kt^1)$.

D3. If $Kt^1 \subset Kt^2$ then Kt^1 is a subclass of Kt^2; if $(Kt^1 \subset Kt^2)$. $\sim (Kt^2 \subset Kt^1)$ then Kt^1 is a proper subclass of Kt^2.

From the definitions we get the statements:

T1. $Kt^1 \cap Kt^2 \subset\supset Kt^2 \cap Kt^1$, $Kt^1 \cup Kt^2 \subset\supset Kt^2 \cup Kt^1$
T2. $Kt^1 \cap Kt^2 \subset Kt^1$, $Kt^1 \cap Kt^2 \subset Kt^2$
T3. $Kt^1 \subset Kt^1 \cup Kt^2$, $Kt^2 \subset Kt^1 \cup Kt^2$
T4. $Kt^1 \cup Kt^1 \subset\supset Kt^1$, $Kt^1 \cap Kt^1 \subset\supset Kt^1$
T5. $Kt^1 \cap Kt^2 \cap Kt^3 \subset\supset Kt^1 \cap (Kt^2 \cap Kt^3)$

T6. $Kt^1 \cup Kt^2 \cup Kt^3 \subset\supset Kt^1 \cup (Kt^2 \cup Kt^3)$

T7. $(Kt^1 \cup Kt^2) \cap Kt^3 \subset\supset (Kt^1 \cap Kt^3) \cup (Kt^2 \cap Kt^3)$

T8. $(Kt^1 \cap Kt^2) \cup Kt^3 \subset\supset (Kt^1 \cup Kt^3) \cap (Kt^2 \cup Kt^3)$

T9. If Kt^1 is empty, then $Kt^1 \cup Kt^2 \subset\supset Kt^2$

T10. If Kt^1 is empty, then $Kt^1 \cap Kt^2 \subset\supset Kt^1$

T11. If Kt^1 is universal, then $Kt^1 \cup Kt^2 \subset\supset Kt^1$

T12. If Kt^1 is universal, then $Kt^1 \cap Kt^2 \subset\supset Kt^2$

T13. $(Kt^1 \subset Kt^2) \rightarrow (Kt^1 \cup Kt^3 \subset Kt^2 \cup Kt^3)$

T14. $(Kt^1 \subset Kt^2) \rightarrow (Kt^1 \cap Kt^3 \subset Kt^2 \cap Kt^3)$

T15. $(Kt^1 \subset\supset Kt^2) \rightarrow (Kt^1 \cup Kt^3 \subset\supset Kt^2 \cup Kt^3)$

T16. $(Kt^1 \subset\supset Kt^2) \rightarrow (Kt^1 \cap Kt^3 \subset\supset Kt^2 \cap Kt^3)$

T17. $(Kt^1 \rightleftharpoons \overline{Kt^2}) \rightarrow (Kt^2 \rightleftharpoons \overline{Kt^1})$

T18. $(Kt^1 \subset Kt^2) \cdot (Kt^2 \subset Kt^3) \rightarrow (Kt^1 \subset Kt^3)$

T19. $(t^1 \rightharpoonup t^2) \rightarrow (Kt^2 \subset Kt^1)$

T20. $(Kt^1 \subset Kt^2) \rightarrow (t^2 \rightharpoonup t^1)$

T21. $Kt \cap \overline{Kt}$ is empty, $Kt \cup \overline{Kt}$ is universal.

T22. If Kt^1 is empty then $Kt^1 \subset Kt^2$ (a consequence of *D4*XV6) for any Kt^2.
If Kt^1 is universal then $Kt^2 \subset Kt^1$ (consequence of *D1*XV7) for any Kt^2.

T23. If Kt^1 and Kt^2 are both empty then $Kt^1 \subset\supset Kt^2$.

T24. If Kt^1 and Kt^2 are both universal then $Kt^1 \subset\supset Kt^2$.

T25. If $Kt^1 \subset Kt^2$ and t^2 is an individual term then $Kt^2 \subset Kt^1$, $Kt^1 \subset\supset Kt^2$.

For classes which contain n-tuples as elements, according to *A1*VIII8 and *A2*VIII8:

T26. $(t^i \rightharpoonup t^k) \rightarrow (K(t^1, \ldots, t^n) \subset K(t^1, \ldots, t^n)(t^k/t^i))$,

where t^k is any of $t^1, \ldots, t^n$

T27. $\sim (Kt^k \subset K(t^1, \ldots, t^n))$, $\sim (K(t^1, \ldots, t^n) \subset Kt^k)$.

10. TERMS

From sentences about classes one forms terms of the following kind:

1) $t^i\downarrow \in Kt$, $t^i\downarrow \urcorner \in Kt$ for "t^i included (not included) in Kt";

2) $(Kt^1)\downarrow \subset Kt^2$, $(Kt^1)\downarrow \urcorner \subset Kt^2$

3) $(Kt)\downarrow PX$, where Kt occurs in X.

According to the rules seen in section seven of Chapter Eight, we form terms of the type

$$t^i \downarrow \in Kt^1 \cdot \in Kt^2,\ t^i \downarrow \in Kt^1 \cdot \neg \in Kt^2,\ (Kt^1) \downarrow \subset Kt^2 \cdot \neg \subset Kt^3,$$

etc. (similarly for the sign : ; there are possible combinations with the signs : and ·).

We accept the definition:

D1. Kt^1 and Kt^2 are disjoint if and only if for any individual t^i

$$((t^i \downarrow \in Kt^1) \neg \in Kt^2) \cdot ((t^i \downarrow \in Kt^2) \neg \in Kt^1).$$

11. THE NUMBER OF ELEMENTS OF A CLASS

Although the theory of numbers is a part of mathematics, it is traditional to develop its foundations in logic. We limit ourselves here to a few remarks relevant to our conception of the logical theory of knowledge.

Numbers are abstract objects of a special type. We are interested here only in those which are formed for the knowledge of classes and which are designated by the signs "zero", "one", "two", etc. (signs of natural numbers), "finite number" ("finite"), "infinite number" ("infinite"). They are formed according to some intuitive premisses which can partially be fixed in the following system of definitions and assertions.

D1. If α is a number such that $Kt \Leftarrow \alpha$ then α is a number of the elements of Kt. The expressions "The number of elements of Kt is α", "The number of elements of Kt is equal to α", etc., are just variants of $Kt \Leftarrow \alpha$.

A1. If Kt is empty then the number of elements is zero (0).

A2. If t is an individual term, then the number of elements of Kt is one (1).

A3. If $Kt^1, Kt^2, \ldots, Kt^n$ $(n \geqslant 2)$ are disjoint, the number of elements of Kt^1 is α^1, the number of elements of Kt^2 is $\alpha^2, \ldots$, the number of elements of Kt^n is α^n, then the number of elements of $Kt^1 \cup Kt^2 \cup \ldots \cup Kt^n$ is $\alpha^1 + \alpha^2 + \ldots + \alpha^n$ (the sum of the numbers $\alpha^1, \alpha^2, \ldots, \alpha^n$).

D2. There is a one-to-one correspondence between Kt^1 and Kt^2 if and only if it is possible to correlate each element of one of them with one and only one element of the other so that pairwise different elements of one correspond to the pairwise different elements of the other.

A4. The number of elements of Kt^1 is equal to the number of elements of Kt^2 (classes Kt^1 and Kt^2 are equinumerous) if and only if Kt^1 and Kt^2 are in one-to-one correspondence. If α is the number of elements of Kt^1 and β of Kt^2 then the fact that Kt^1 and Kt^2 are equinumerous is written as $\alpha = \beta$.

A5. The number of elements of Kt^1 is larger than the number of elements of Kt^2 if and only if no matter which proper subclass Kt^3 of Kt^1, which is in one-to-one correspondence with Kt^2, one chooses there will always be an element of Kt^1 which will correspond to no element of Kt^2. If α is the number of elements of Kt^1 and β of Kt^2 then "α is greater than β" will be written as $\alpha > \beta$.

D3. $\alpha < \beta \equiv \beta > \alpha$.

A6. The number of elements of a class is finite in the following cases (and only in them):

1) if the number of elements of Kt is equal to one, then the number of elements of Kt is finite;

2) if $Kt \subset\supset Kt^1 \cup Kt^2$ and the number of elements of Kt^1 is finite and the number of elements of Kt^2 is finite, then the number of elements of Kt is finite.

A7. The number of elements of Kt is infinite if and only if it is impossible to construct Kt^1 and Kt^2, each of which has a finite number of elements and $Kt \subset\supset Kt^1 \cup Kt^2$.

From the definitions and assertions accepted above, we obtain:

T1. $(\alpha = \beta) \leftrightarrow \sim(\alpha > \beta) \cdot \sim(\beta > \alpha)$

T2. $(Kt^1 \subset Kt^2) \rightarrow \sim(\alpha > \beta)$,

where α is the number of elements of Kt^1 and β of Kt^2.

T3. $(Kt^1 \subset\supset Kt^2) \rightarrow (\alpha = \beta)$.

T4. If α^1 is the number of elements of Kt^1, α^2 of Kt^2, α^3 of Kt^3, α of Kt, then

1) $(\alpha^1 + \alpha^2) = (\alpha^2 + \alpha^1)$
2) $(\alpha^1 + \alpha^2 + \alpha^3) = (\alpha^1 + (\alpha^2 + \alpha^3))$
3) $(\alpha + 0) = \alpha$
4) $(\alpha^1 = \alpha^2) \rightarrow ((\alpha^1 + \alpha^3) = (\alpha^2 + \alpha^3))$.

T5. If $Kt^1 \subset \supset Kt^2$, then there is a one-to-one correspondence between Kt^1 and Kt^2.

T6. If one of Kt^1 and Kt^2 is a proper subclass of the other and finite, then there is no one-to-one correspondence between them.

But there is a one-to-one correspondence between a class and its proper subclass if both are infinite (e.g., in the case of the class of all natural numbers and of the even numbers). Therefore, the following assertion is wrong in the general case: if Kt^1 is a proper subclass of Kt^2 then the number of elements of the first is smaller than the number of elements of the second. Also incorrect is: if the number of elements of Kt^1 and that of Kt^2 is infinite, then the two classes are equinumerous.

The number of elements of classes is also fixed by the expressions "many", "few", "very many", "very few", etc. (these are indeterminate or archaic numbers). Of interest are such numbers of the above type which could be called practically infinite: these are cases where the number of elements of the class is so great (even though finite) that it can be taken as infinite.

12. COMPOSITION AND POWER OF A CLASS

D1. To explain the composition of a class means to explain which individuals compose it. To explain the existential (potential) composition of a class means to explain which existing (possible) individuals are its elements.

D2. The power of a class is the number of its elements. The existential (potential) power of a class is the number of existing (possible) individuals which are its elements.

If the investigator has to explain the composition and power of a given class, this is always done at a fixed time. The time is not always that indicated in the definition of the class (e.g., "Russian authors of the 19th Century") but the time when the investigator undertakes activities to establish the composition and power of the class.

A class can be formed so as to include only individuals existing at that time. Thus, classes can be variable and constant both in regard to composition and in regard to power:

1) constant in a given time-interval, if in every moment of that time-interval the investigation explaining its composition obtains the same

result; similarly for power; constant in general if at any time the investigator will arrive at the same result;

2) variable during a given time-interval, if at one time in this interval the investigator arrives at one result, at another he arrives at another result since some elements can come to be and others disappear; the number of elements of a class may vary in time; variable in general, if variable for any time-interval.

3) the most diverse combinations of 1 and 2 are possible; e.g., some elements of a given class disappear while new ones appear and the number of elements grows.

In addition to the distinction between finite and infinite classes there is one between limited and unlimited classes. A class can be formed in such a way that the number of its elements is infinite but limited in the sense that no new elements will appear in time. On the other hand, the number of elements of a class may be finite even though we do not limit it.

13. FUNCTIONS

D1. Class Kt^2 corresponds to individual t^1 if and only if each of the elements of class Kt^2 corresponds to individual t^1.

D2. Individual t^1 corresponds to class Kt^2 if and only if individual t^1 corresponds to any (each) element of class Kt^2.

D3. Class Kt^2 corresponds to class Kt^1 if and only if to every element of class Kt^1 corresponds some non-empty subclass of class Kt^2.

D4. If we define a way to establish for every individual of a class Kt^1 which subclass of Kt^2 corresponds to it then we will say that we have defined (found, known, etc.) a type (species, way) of correspondence of class Kt^2 to class Kt^1 or that we have defined a function from t^1 to t^2. We will use for this

$$t^2 \Leftarrow f(t^1),$$

where f indicates the mode of correspondence.

D5. Individuals from the value-range of t^1 are arguments and individuals from the value-range of t^2 are functionals of the given function.

D6. Kt^1 is the range of the definition of $t^2 \Leftarrow f(t^1)$ and Kt^2 is its value-range.

D7. We will call the realization of $t^2 \Leftarrow f(t^1)$ the discovery of the functionals $t^{i1},\dots, t^{im}$ $(m \geqslant 1)$ for the given argument t^k. We designate it as

$$(t^2 = t^{i1},\dots, t^{im}) \Leftarrow f(t^1 = t^k).$$

D8. Two functions $t^2 \Leftarrow f^1(t^1)$ and $t^4 \Leftarrow f^2(t^3)$ are identical if and only if

1) $Kt^1 \subset\supset Kt^3$
2) $Kt^2 \subset\supset Kt^4$
3) for each argument t^k

$$(t^2 = t^{i1},\dots, t^{im}) \Leftarrow f^1(t^1 = t^k)$$
$$(t^4 = t^{i1},\dots, t^{im}) \Leftarrow f^2(t^3 = t^k).$$

Otherwise they are different.

D9. The function $t^1 \Leftarrow f^1(t^2)$ is called inverse relative to $t^2 \Leftarrow f^2(t^1)$.

If Kt^1 and Kt^2 are given, then the character (type) of f is determined by the properties of these classes and by the purpose of the investigation. In particular, the type of f can depend on the definitions, the results of observation, etc. We should note the following here: the stipulation $t^2 \Leftarrow f(t^1)$ means only what was indicated in *D4*; the reasons for accepting it, however, are not taken into consideration; and in this sense it is not a sentence about empirical connections of objects or about logical connections of terms. It is an autonomous logical form. In particular, we can establish functions where the arguments are temporal or spatial intervals which in themselves are not causes of events or premisses of reasonings.

In the case of $t^1 \Leftarrow f(t^2)$ the terms t^1 and t^2 can be n-tuples $(n \geqslant 2)$ of terms. We note that these terms do not depend on each other for meaning; similarly, the individual terms of their value-ranges do not depend on each other for meaning.

Sentences which formulate functions are sentences with many-place predicates. Everything said about the latter applies to the former. But they also have their own peculiar properties conditioned by the properties of their predicates (i.e., of the types of functions).

14. FUNCTIONS WITH SENTENCES

Let $X^1,\dots, X^n$, Y be any sentences and $\downarrow X^1,\dots, \downarrow X^n, \downarrow Y$ the corresponding terms. Let $\downarrow X_i^1 \in K \downarrow X^1, \ldots, \downarrow X_i^n \in K \downarrow X^n, \downarrow Y_i \in K \downarrow Y$. If we have the function

$$\downarrow Y \Leftarrow f(\downarrow X^1, \ldots, \downarrow X^n),$$

then an expression of the type

$$\downarrow Y_i \Leftarrow f(\downarrow X_i^1, \ldots, \downarrow X_i^n)$$

is its realization.

$$A1.\ (\downarrow Y \Leftarrow f(\downarrow X^1, \ldots, \downarrow X^n)) \cdot X_i^1 \cdot \ldots \cdot X_i^n \rightarrow Y_i$$

holds for sentences about such functions.

15. DEFINITIONS

Science contains a large number of definitions of the type

$$[Q\gamma^k\gamma_k] \Leftarrow f([P^1\gamma^1\gamma_1], \ldots, [P^n\gamma^n\gamma_n])$$
$$[s^* \leftarrow Q\gamma^k\gamma_k] \Leftarrow f([s^* \leftarrow P^1\gamma^1\gamma_1], \ldots, [s^* \leftarrow P^n\gamma^n\gamma_n]),$$

where $Q\gamma^k\gamma_k$ are the newly introduced terms and

$$[\gamma^k] \Leftarrow f([\gamma^1], \ldots, [\gamma^n]).$$
$$A1.\ ([P_1] \Leftarrow f([P_2], \ldots, [P_n])) \leftrightarrow ([s^* \leftarrow P_1] \Leftarrow f([s^* \leftarrow P_2], \ldots$$
$$\ldots, [s^* \leftarrow P_n]).$$

holds for such definitions.

Usually the predicates $Q\gamma^k\gamma_k$ are distributed into parts and the definitions appear as definitions of Q and γ_k (e.g., the expressions "speed", "force", "km/h", etc. are considered autonomous units of the language of science). However, the expressions Q and γ_k have meaning (and play the part of autonomous signs) only to the extent that it is assumed that they are parts of $Q\gamma^k\gamma_k$.

16. MODELS

Let our task be to investigate the objects of some class Kt^1 (it can also be an individual object), i.e., to obtain about these objects some sentences meeting certain requirements. This can be done in two ways:

1) one can investigate the very representatives of this class of objects (the object itself) Kt^1;

2) one selects (or establishes, in particular) some other objects of a class Kt^2 which are investigated in place of the objects of Kt^1 and then from the sentences obtained here one obtains according to certain rules sentences relative to the objects of Kt^1.

D1. The objects of Kt^1 are object-originals relative to the objects of Kt^2 and the objects of Kt^2 are object-models relative to the objects of Kt^1.

Object-models are selected so that

$$\downarrow Y^1 \Leftarrow f^1(\downarrow X^1), \ldots, \downarrow Y^n \Leftarrow f^n(\downarrow X^n),$$

where $X^1, \ldots, X^n$ are sentences obtained from investigation of the models and $Y^1, \ldots, Y^n$ are sentences relating to object-originals and satisfying some previously established requirements. The functions $f^1, \ldots, f^n$ are the rules of substitution of terms relative to object-models for terms relative to object-originals. The obtaining of Y^i from X^i according to these rules has nothing in common with reasoning by analogy since the models are selected here in such a way that the rules apply. It is clear that there has to be some previous knowledge about object-originals and object-models so that there is some *a priori* conviction on the possibility and suitability of $Y^1, \ldots, Y^n$. Failures in such instances do not change matters.

17. LOGIC OF CLASSES

Alphabet:

1) $\in$ is a two-place predicate constant (inclusion of an individual in a class);

2) $\subset$ is a two-place predicate constant (inclusion of a class in a class);

3) K is a class-generative operator.

D1. If a is a subject, then Ka is a subject.

The definition *D1* is an addition to the definition of a subject in the theory of terms. And the logic of classes is, from this point of view, a theory of terms of the type Ka.

D2. $\in(a, Kb)$ and $\subset(Ka, Kb)$ are elementary formulae of the logic of classes if and only if a and b are subjects. In what follows we will use the symbols $a \in Kb$ and $Ka \subset Kb$.

D3. An elementary formula of the logic of classes is a K-formula.

D4. A formula of the logic of classes is a K-formula in which an elementary formula of the logic of classes occurs.

Axioms *AI*:

1. $(s^3 \in Ks^1) \wedge (s^3 \in Ks^2) \dashv\vdash s^3 \in K(s^1 s^2)$
2. $(s^3 \in Ks^1) \vee (s^3 \in Ks^2) \dashv\vdash s^3 \in K(s^1 \vee s^2)$
3. $(s^1 \in K \sim s^2) \vdash \sim (s^1 \in Ks^2)$
4. $(\exists s^1)(s^1 \in Ks^2) \vdash (\exists s^2)(s^2 \in Ks^1)$
5. $(s^1 \in Ks^2)(\forall s^2)(s^2 \in Ks^3) \vdash (s^1 \in Ks^3)$
6. $(Ks^1 \subset Ks^2) \dashv\vdash (\forall s^1)(s^1 \in Ks^2)$

R1. Substitution of $\in$ and $\subset$ for two-place predicate variables.

If the logic of classes is constructed independently of the theory of terms the following substitution rule *R2* is adopted:

1) substitution of s^1 for any s^2 (wherever s^2 occurs in the formula);
2) substitution of $\sim \sim s$ by s and vice versa;
3) substitution of $s^2 s^1$ for $s^1 s^2$;
4) substitution of ss for s and vice versa;
5) substitution of $s^1(s^2 s^3)$ for $s^1 s^2 s^3$ and vice versa;
6) substitution of $\sim s^1 \vee \sim s^2$ for $\sim (s^1 s^2)$ and vice versa.

The axioms 1–5 form the syllogistic of classes.

The connection between the syllogistic of classes and the classical predicate syllogistic is established in *AII*:

1. $P(s) \dashv\vdash s \in K(sc \downarrow P)$
2. $s^1 \in Ks^2 \dashv\vdash (Pc \downarrow s^2)(s^1)$

In this case the syllogistic of classes can be obtained from the classical predicate syllogistic and vice versa.

If we accept the axioms *AIII*

$$\vdash (s^1 \rightharpoonup s^2) \leftrightarrow (\forall s^2)(s^2 \in Ks^1)$$

then the axioms 2, 3, 5, 6 and 7 of the group *AI* turn out to be dependent in the theory of terms, expanded at the expense of a proper part of the logic of classes.

Axiomatic schemata *AIV*:

1. $(\forall s^1)\, X \cdot (s^2 \in Ks^1) \vdash Y$,

where Y is formed from X by substitution of s^2 for s^1 wherever s^1 freely occurs in X.

2. $Y \cdot (s^2 \in Ks^1) \vdash (\exists s^1)\, X$.

18. QUASI-CLASSICAL CASES IN THEORY OF QUANTIFIERS

The assumption that the value-ranges of all individual variables are identical can be written, accepting the axiom A^*, as

$$\vdash s^1 \in K s^2 .$$

The formulae

$$\vdash (\forall s^1)(s^1 \in K s^2)$$
$$\vdash (\forall s^1)(\forall s^2)(s^1 \in K s^2)$$
$$\vdash s^1 \rightharpoonup s^2$$

are equivalent to it.

The axiom A^* is an explication of the assertions

$$(\forall s^1)\, P(s^1) \supset P(s^2)$$
$$P(s^2) \supset (\exists s^1)\, P(s^1),$$

which are basic to the axioms of classical (and intuitionist) predicate logic.

It is easy to show that, thanks to A^*, the formulae

$$(\forall s^1)\, P(s^1) \rightarrowtail P(s^2)$$
$$P(s^2) \rightarrowtail (\exists s^1)\, P(s^1),$$

corresponding to the formulae of classical logic introduced above, are provable.

CHAPTER FIFTEEN

EXISTENTIAL LOGIC

1. NON-CLASSICAL CASES

The non-classical theory of existence is formed by the following additions to the previously adduced systems.

Alphabet:

1) E is the predicate constant "exists";
2) U is the predicate constant "universal";

D1. E and U are predicates.

D2. $\alpha E(a)$ and $\alpha U(a)$ are elementary existential formulae if and only if a is a subject and α indicates the presence of $\neg$ or of ? or the absence thereof. Elementary existential formulae are elementary subject-predicate formulae.

Axioms *AI*:

1. $\vdash E(s^1, \ldots, s^n) \leftrightarrow E(s^1) \cdot \ldots \cdot E(s^n)$
2. $\vdash \neg E(s^1, \ldots, s^n) \leftrightarrow \neg E(s^1) \vee \ldots \vee \neg E(s^n)$
3. $\vdash E(s) \neg E(s \downarrow P) \rightarrow E(s \neg \downarrow P)$
4. $\vdash E(s)?E(s \downarrow P) \rightarrow E(s? \downarrow P)$
5. $\vdash U(s) \rightarrow E(s)$
6. $\vdash U(s) \leftrightarrow \neg E(\sim s)$
7. $\vdash \neg U(s) \leftrightarrow E(\sim s)$
8. $\vdash (\exists s)\, E(s) \rightarrow E(s)$
9. $\vdash (\neg \exists s)\, E(s) \leftrightarrow \neg E(s)$
10. $\vdash \neg E(\sim ss)$

Axiomatic Schemata *AII*:

1. $\vdash X \rightarrow E(\downarrow X)$
2. $\vdash U(\downarrow X) \rightarrow X$
3. $\vdash (\exists a)\, X \leftrightarrow E(a \downarrow X)$
4. $\vdash (\neg \exists a)\, X \leftrightarrow \neg E(a \downarrow X)$
5. $\vdash E(a \downarrow X) \rightarrow Ea$ where E does not occur in X.

$$\left.\begin{array}{l}6.\ \vdash \neg E(a)\rightarrow\neg E(a\downarrow X)\\ 7.\ \vdash ?E(a)\rightarrow\sim E(a\downarrow X)\end{array}\right\}\text{where } E \text{ does not occur in } X.$$

8. $\vdash E(\downarrow X)\leftrightarrow E(a\downarrow X)$, where a freely occurs in X.

Axiomatic Schemata *AIII*:

1. $\vdash E(\downarrow(X \vee Y))\leftrightarrow E(\downarrow X) \vee E(\downarrow Y)$
2. $\vdash \neg E(\downarrow(X \vee Y))\leftrightarrow\neg E(\downarrow X)\neg E(\downarrow Y)$
3. $\vdash E(\downarrow(XY))\rightarrow E(\downarrow X)E(\downarrow Y)$
4. $\vdash \neg E(\downarrow X)\vee\neg E(\downarrow Y)\rightarrow\neg E(\downarrow(XY))$
5. $\vdash U(\downarrow X)E(\downarrow Y)\rightarrow E(\downarrow(XY))$
6. $\vdash U(\downarrow(X \vee Y))\rightarrow U(\downarrow X)\vee E(\downarrow Y)$
7. $\vdash E(\downarrow X)?E(\downarrow Y)\rightarrow ?E(\downarrow(XY))$
8. $\vdash ?E(\downarrow X)?E(\downarrow Y)\rightarrow ?E(\downarrow(XY))$

Inference Rule:
R1. If $\vdash x$, then $\vdash U(\downarrow x)$.

2. CLASSICAL CASES

To obtain the classical case it is sufficient to eliminate the axioms *AI2 AI8, AI10, AII4, AIII2, AIII4, AIII7,* and *AIII8,* and to put $\sim$ for $\neg$ throughout.

3. INTERPRETATION

We adduce the interpretation:

1) subjects take the values v^1 and nv^1;

2) $E(a^1,\ldots, a^n)$ is equivalent to $E(a^1)\cdot\ldots\cdot E(a^n)$;

3) $x\rightarrow y$ is equivalent to $x\supset y$;

4) x, $\downarrow x$ and $a\downarrow x$ are equivalent;

5) if the values of b and c are equivalent, then those of $\alpha E(b)$ and $\alpha E(c)$ are equivalent;

6) if Ea and $\alpha E(a\downarrow Q)$ have the value v^1, then $E(a\alpha\downarrow Q)$ has the value v^1;

7) $U(a)$ is equivalent to $\neg E(\sim a)$, $\neg U(a)$ to $E(\sim a)$;

8) $(\alpha\exists a)\,E(a)$ is equivalent to $\alpha E(a)$, $(\alpha\exists a)X$ to $\alpha E(a\downarrow X)$;

9) if Ea has the value $v^1(nv^1)$, then a can (not) take the value v^1; vice versa; if a cannot take the value v^1, then $?E(a)$ has the value nv^1.

*T*1.* All formulae provable in the adduced system are tautologies.

CHAPTER SIXTEEN

MODAL SENTENCES

1. EVENTS

D1. If X is a sentence, then the term $\downarrow X$ is a term of the event, about which X talks. The symbol $\downarrow X$ reads "The fact that X", "That which X", etc. For example, the event talked about in "The earth revolves around the sun" is designated by the term "The fact that the earth revolves around the sun". Clearly the relation between $\downarrow X$ and $\downarrow \sim X$ is determined by that between X and $\sim X$.

In what follows we will use simpler symbols $x, y, \ldots$ for $\downarrow X, \downarrow Y, \ldots$, if the sentential symbols are not broken down into parts.

D2. $x \leftarrow E \equiv [X] \leftarrow v^1$

$x \urcorner \leftarrow E \equiv [X] \urcorner \leftarrow v^1.$

Events exist or do not exist in a given situation or in any situation. The latter can be clearly given or presupposed (clear from the context). It can be given: 1) by reference to a region of space; 2) by reference to time; 3) by listing a set of events; 4) by a combination of 1)–3). We will say that the coordinates of an event are given if 1)–4) are used to give the situation in which it exists. If, for some reason, the coordinates are indifferent, the fixing of this circumstance is a special case of the fixing of the coordinates.

D3. Event x will be called local if X is true in some coordinates and not true in others, and universal if X is true in any coordinates. Local events form the object-range, for which modal signs are designed.

In order not to complicate things we will not use special signs for the coordinates of events. But we assume that if need be they can be written according to the following stipulations:

1) to each sentence one can assign a sign of the coordinates of the event which is talked about in this sentence;

2) within the limits of one and the same assertion we assume identity of coordinates for all events which are talked about in this assertion so that one can assign identical signs of coordinates to all sentences found

in the assertion (or the expressions "in the same coordinates" and "for the same coordinates" can be assigned to the assertion as a whole).

2. BASIC MODALITIES

D1. Sentences on the possibility and necessity of occurrence of events, on their intrinsic negations, and on the indeterminate forms and all possible extrinsic negations, are the basic modal propositions. We will represent them by the symbols:

1) Mx for "x is possible", "The occurrence of x is possible", "x can happen", etc.;

2) $\urcorner\, Mx$ for "x is impossible";

3) $?Mx$ for "it cannot be established whether Mx or $\urcorner\, Mx$";

4) Nx for "x is necessary", "x necessarily happens", "the occurrence of x is unavoidable";

5) $\urcorner\, Nx$ for "x is non-necessary";

6) $?Nx$ for "it cannot be established whether Nx or $\urcorner\, Nx$".

The expression "it cannot be established" means that in view of the conditions obtaining in a given domain of science we have no basis for assigning the event x to the possible (necessary) events and we have no basis for assigning this event to impossible (non-necessary) events.

The signs M and N are predicates: the symbols

$$\alpha Mx \quad \text{and} \quad \alpha Nx$$

are fully adequate to the symbols

$$x\alpha \leftarrow M \quad \text{and} \quad x\alpha \leftarrow N.$$

Moreover the signs M and N are predicates which characterize the events themselves and not the physical states of those who speak about the events. They are structural elements of knowledge and as such they are free of psychological overtones. The sentences which contain them have meaning and are subject to verification independent of the psychic states of those who propose them: they have objective meaning and a certain truth-value.

The formation of modal sentences with the adduced symbols is, of course, their schematization and standardization. In real languages the modal signs can occur in other positions. For example, in the sentence "*s* can have *P*" the word "can" is situated in such a way that its character of predicate is hidden; but knowing the rules of English, we can justifiably replace this sentence with "*s* which has *P* is possible".

As we see it, modal sentences are sentences about modal events. One sometimes distinguishes this from the modality of sentences and studies expressions like "*X* is a possible (necessary, etc.) sentence". But in fact the modality of sentences reduces one way or another to the modality of events. Thus in explaining the sense of the expression "*X* is a necessary sentence" we ultimately arrive at the conclusion either that we can do without the term "necessary" (since, for example, we find that *X* is always true, i.e., tautological) or that what is meant is the necessity of the event *x*.

One must distinguish the modality of events from the modality of the search for events. The latter is also an event but its modality does not always coincide with that of the event which is sought. The sole rule which can be formulated here is that it is impossible to find impossible events.

3. INTRODUCTION OF MODALITY

The introduction of modal predicates for use in a given domain of science depends on the conditions and needs of this science. If the need is felt then it has to be satisfied according to the following logical principles.

In the introduction of the predicate "possible" one can meet with three cases: 1) some events are recognized as possible (impossible), i.e., they are listed in the class of possibles (impossibles); the class of impossible (possible) events here remains indeterminate; 2) some of the events are recognized as possible, others as impossible and still others fall into neither class; 3) every event falls to either the possible or the impossible. Similarly for the predicate "necessary". Thus, the expression "The event is not possible" can mean that the event is not counted among the possible, or it can mean that the event is impossible. And these are not the same. Similarly for necessity. We distinguish these through the distinction between intrinsic and extrinsic negation: $\sim Mx$ and $\urcorner Mx$, $\sim Nx$ and $\urcorner Nx$. If an event does not fall among the possible (necessary) and among the impossible (unnecessary) then it is indeterminate from the point of view of possibility (necessity). This indeterminacy is not once and for all given. Conditions in a given science can change and modally indeterminate events can become determinate. One cannot *a priori* exclude events which remain indeterminate no matter how the science changes.

The assertions

$$Mx : \neg Mx : ?Mx$$
$$Nx : \neg Nx : ?Nx$$

hold in the introduction of every modal predicate (or in the establishment of the modal character of events).

If indeterminacy is excluded, i.e., if $\sim ?Mx$ and $\sim ?Nx$, then

$$Mx : \neg Mx$$
$$Nx : \neg Nx .$$

If one introduces both predicates M and N then the following have to be satisfied:

$$Nx \vdash Mx$$
$$\neg Mx \vdash \neg Nx$$
$$\sim Mx \vdash \sim Nx .$$

If an event is neither in the class of possibles nor in the class of impossibles, this does not mean that it is neither in the class of necessaries nor in the class of non-necessaries, i.e., in the general case

$$\sim (?Mx \vdash ?Nx) .$$

The assertions

$$Mx : M \sim x$$
$$Nx : N \sim x$$
$$Nx \vee N \sim x$$

do not always hold since there are cases where

$$Mx \cdot M \sim x$$
$$\neg Nx \cdot \neg N \sim x .$$

For necessity always

$$\sim (Nx \cdot N \sim x) .$$

Let the symbols

$$Mod, Mod^1, Mod^2, \ldots$$

singly be any of αN and αM, and a difference of superscripts indicate only that these signs can be taken in different combinations. We adopt the definition:

D1. Events x^1 and x^2 are modally dependent if

$$(Mod^1 x^1 \vdash Mod^2 x^2) \vee (Mod^3 x^2 \vdash Mod^4 x^1).$$

and modally independent if

$$\sim (Mod^1 x^1 \vdash Mod^2 x^2) \cdot \sim (Mod^3 x^2 \vdash Mod^4 x^1).$$

4. THE LOGICAL LIMITS OF MODALITY

Which events are possible, necessary, etc., within a given science are determined in that science. But logic establishes the limits beyond which no science can go in the definition of the classes of possible, necessary, etc., events. These limits represent the so-called logical modalities. If the modality (possibility, necessity, etc.) of events is explained through concrete investigation in a given science then it is called factual or empirical.

Of the various ways of defining logical modality contemporary logic favors two: 1) logical modalities are defined through satisfiability, general validity (tautologousness), consistency, etc., of the formulae of matrix construction (algebra of logic, usually two-valued); 2) logical modalities are defined through the provability, unprovability, etc., of formulae of a logical calculus (classical or other, with strict or strong implication, etc.).

Let the symbols

$$LN \quad \text{and} \quad LM$$

read respectively "logically necessary" and "logically possible". We accept the assertions:

A1. $LN \sim (x \cdot \sim x)$
A2. $(X \vdash Y) \cdot LNx \rightarrow LNy$
A3. $LMx \equiv \urcorner LN \sim x$
A4. $\urcorner LMx \equiv LN \sim x$.

We also adopt the definitions:

D1. An event is logically necessary only in function of *A1* and *A2* (otherwise it is non-necessary).

D2. An event is logically possible only in function of *A3*. It is logically impossible only in function of *A4*.

The relations between logical modality and modality in general are defined by the assertions:

A5. $LNx \vdash Nx$
A6. $Mx \vdash LMx$
A7. $\sim(Nx \vdash LNx)$
A8. $\sim(LMx \vdash Mx)$.

5. PREDICTION

D1. Let it be the case that in time δ^1 event x does not exist. A sentence that x will happen in the future at some (specified or unspecified) time after δ^1 will be called a sentence about the future event x or the prediction of x.

Let W be the prediction of x, said at time δ^1. Let δ^2 be a time after δ^1, indicated in W (this can be an indeterminate "will" or a specified "after such and such a time", etc.). One has to distinguish the question about the truth-value of W in δ^1 from that about its truth-value in δ^2.

The first question is solved by the definitions:

D2. W is true in δ^1 if and only if there is a true Y such that $Y \rightarrow W$.

D3. W is false in δ^1 if and only if there is a true Z such that $Z \rightarrow \sim W$.

D4. W is indeterminate in δ^1 if and only if there is no Y and Z such as indicated in *D2* and *D3*.

The second question is solved by the definitions:

D5. W is true in δ^2 (is fulfilled) if and only if $x \leftarrow E$ in δ^2.

D6. W is false in δ^2 (is not fulfilled) if and only if $x \urcorner \leftarrow E$ in δ^2.

In the ideal case there is the following connection between the truth-value of W in δ^1 and its truth-value in δ^2:

1) if W is true in δ^1 then it will also be true in δ^2; if W is true in δ^2 then it cannot be false in δ^1;

2) if W is false in δ^1 then it will be false in δ^2; if W is false in δ^2 then it cannot be true in δ^1.

But, these values are established independently of one another. What is more, the truth (falsity) of W in δ^1 does not follow from its truth (falsity) in δ^2; it could be indeterminate.

When one says that someone who predicted an event was right in his time one commits a logical error. One can only say that this someone proved to be right (that the prediction was fulfilled). And this is a completely different matter. In many cases predictions are only indeterminate in their time and their base (i.e., the discovery of Y) proves illusory.

In this regard the famous "paradox" of the foreigner offers no difficulty: the foreigner comes into a country where every foreigner has to enunciate a sentence and is executed for a false sentence (the foreigner enunciates "You will execute me").

All that has been said about predictions can be, with some adjustment, applied to sentences about past events (δ^2 precedes δ^1).

6. THE MEANING OF MODAL PREDICATES

There are different methods of introducing modal predicates. For example, if an event is observed once, it is considered possible; if in our experience an event happens in a sufficient number of cases, then it is considered necessary; if X follows from the definition of t, then x is considered necessary and $\sim x$ impossible. The general schema has the form "If Y then $Mod\,x$", where Y is some set of items known. This schema is no guarantee that $Mod\,x$ will be true. This is in short how people act in introducing modal signs.

In order to answer the question on the meaning of the terms M and N it is necessary to question the investigator who uses these terms as to the following: why does he consider one or another event possible, necessary, etc. It is not hard to see that in most cases where modal signs are used one can (in principle) get along without them: they are reduced to other signs. Thus, in some cases $M\,(s \leftarrow PX)$ is replaced by $(\exists s)\,X$. However, there are cases where the modal signs are irreplaceable since they fulfill a specific function.

They have this special function above all when one has to do with prognoses relative to future events, the occurrence or non-occurrence of which depend on a set of circumstances. The general schema for introducing modal predicates for such cases can be written as:

Let it be the case that we know: 1) if event $y^1 \cdot \ldots \cdot y^n$ happens then (after it, after some time, etc.) event x happens; 2) if event $z^1 \cdot \ldots \cdot z^m$ happens then event x does not happen. More simply: 1) if $Y^1 \cdot \ldots \cdot Y^n$, then X; 2) if $Z^1 \cdot \ldots \cdot Z^m$, then $\sim X$. Let the first be V and the second W. Each of $y^1, \ldots, y^n$ will be called a circumstance favoring the occurrence of x and each of $z^1, \ldots, z^m$ will be one hindering it.

In order to carry out prognoses in a scientific way (not as presuppositions or guesses) the following are needed: 1) knowledge of the type of V and W; 2) knowledge on the present state of affairs (designated by U). We call V and W the criteria of the prognoses and U the base thereof. The character of the prognoses depends on the character of V, W and U.

If the criteria of a prognosis are given then its character depends on its base, U. A variety of cases are possible (even under the assumption that we can provide a complete and correct description of a given situation): 1) all favorable circumstances are present; 2) all unfavorable circumstances are present; 3) there are some favorable and none of the unfavorable circumstances; 4) there are some unfavorable, none of the favorable; 5) there are some favorable and some unfavorable. The only case which is logically excluded is where all favorable and all unfavorable would be present, i.e.,

$$\sim (Y^1 \cdot \ldots \cdot Y^n \cdot Z^1 \cdot \ldots \cdot Z^m).$$

In carrying out prognoses one, of course, needs some signs which would formulate in generalized fashion the character of the base of the prognosis. The modal signs play this role: 1) if all favorable circumstances are present then the event is necessary (in these coordinates); 2) if all unfavorable circumstances are present, then the event is impossible; 3) the other, intermediate variants (different combinations of favorable and unfavorable) issue in other possible variants of the modal evaluation of events.

We should note that the above schema does not guarantee true results in all cases. It is a schema for the introduction of modal signs and no more. In a sufficiently large number of cases it has given satisfactory results and this is an empirical fact. This justifies the risk and in any case discloses the meaning of modal signs. The truth of the results depends in each case on the exactness of the criteria of the prognosis, on their efficacity, on the completeness and correctness of the base, on experience in making prognoses, etc.

We have considered the ideal schema. In practice things are much more complex. One and the same circumstance can fall among the favorable and unfavorable; there can be more than two criteria with complex relations between them; the situation itself can change; the base can be too narrow; there can be assumptions, etc. But throughout it is the case that modal predicates are abbreviated designations of certain types of bases of prognosis relative to corresponding criteria. As in the case of E, it is impossible to select objects which correspond to M, N, etc. But one can indicate the type of base and of criteria of the prognosis which is talked about in a sentence with M, N, etc.

Modal evaluations also apply to past and present events. But this has sense only in retrospect: we take ourselves back to when and where the events in question have not yet happened; we find criteria; and depending on the base of the prognosis, we obtain a modal evaluation of the events. This explains the fact that not all events which exist or existed are considered necessary and not all events which did not occur in the past are considered impossible. However, such a modal evaluation of past and present events does not play an important role in science.

7. THE MODALITY OF INDIVIDUAL AND RECURRENT EVENTS

The search for criteria of prognosis is the construction of knowledge on classes of events irrespective of whether we have to do with recurrent or individual events. The latter are generally regarded (often implicitly) as representatives of some classes. There is a difference only in the conditions attendant on the elaboration of the criteria of prognosis: in the first case, the recurrence aids knowledge; in the second, one has to compensate for non-recurrence (through analogy, wider generalization, etc.).

8. THE LOGICAL PROPERTIES OF MODAL PREDICATES

Modal predicates have properties which any other predicates have. In particular, the following hold:

$$\sim \alpha Mx \dashv\vdash \beta Mx \vee \gamma Mx,$$

where α, β and γ are distinct;

$$\sim \alpha Nx \dashv\vdash \beta Nx \vee \gamma Nx$$
$$\vdash \sim (\alpha Mx\, \beta Mx),$$

etc.

In the classical case, when indeterminacy is excluded we accept

$$\sim Mx \dashv\vdash \neg Mx$$
$$\sim \neg Mx \dashv\vdash Mx;$$

are true; similarly for N. But

$$\sim(\sim\urcorner Mx \vdash Mx)$$
$$\sim(\sim\urcorner Nx \vdash Nx)$$

in the general case.

The specific properties of the modal predicates M and N are defined by the following assertions:

A1. The predicates M and N are logically interchangeable, i.e.,

$$Nx \dashv\vdash \urcorner M \sim x$$
$$\urcorner Nx \dashv\vdash M \sim x$$
$$?Nx \dashv\vdash ?M \sim x.$$

A2. $Nx \vdash X$

$X \vdash Mx$

T1. $Nx \vdash Mx.$

Two remarks have to be made before we formulate assertions for events $\downarrow(X^1\cdot\ldots\cdot X^n)$ and $\downarrow(X^1:\ldots:X^n)$. One has to distinguish the expressions $\downarrow(X^1\cdot\ldots\cdot X^n)$ and $\downarrow(X^1:\ldots:X^n)$ from $\downarrow X^1\cdot\ldots\cdot\downarrow X^n$ and $\downarrow X^1:\ldots:\downarrow X^n$. Only the first are (by stipulation) terms of events; the others are terms derived from terms of events. They do not coincide as to meaning. We will have in mind only the first. Further, it is intuitively clear that the assertions

$$Mx^1\cdot\ldots\cdot Mx^n \vdash M(x^1\cdot\ldots\cdot x^n)$$
$$\urcorner Nx^1:\ldots:\urcorner Nx^n \vdash \urcorner N(x^1:\ldots:x^n)$$

are not always valid. This imposes well-known restrictions which influence the character of the assertions below.

A3. $NxMy \vdash M(xy)$

$M(x \vee y) \dashv\vdash Mx \vee My$

$\urcorner M(x \vee y) \dashv\vdash \urcorner Mx \urcorner My$

$?M(x \vee y) \dashv\vdash (?Mx \vee ?My) \sim Mx \sim My$

$\urcorner Mx \vee \urcorner My \vdash \urcorner M(xy)$

$M(xy) \vdash MxMy$

$(X \rightarrow \sim Y) \vdash \urcorner M(xy)$

$M(xy) \vdash \sim(X \rightarrow \sim Y)$

T2. $NxNy \dashv\vdash N(xy)$

$Nx \vee Ny \vdash N(x \vee y)$
$Nx \vee My \vdash M(x \vee y)$
$Mx?My \vdash ?M(xy)$
$?Mx?My \vdash ?M(xy)$
$(?Mx \vee ?My) \sim Mx \sim My \vdash ?M(xy)$

9. RANDOMNESS

Designations:
1) Cx for "x is random";
2) $\neg Cx$ for "x is not random";
3) $?Cx$ for "It is impossible to establish Cx or $\neg Cx$".

The properties of the predicate C are defined by:

A1. $Cx \dashv\vdash Mx \cdot M \sim x$
A2. $\neg Cx \dashv\vdash Mx \cdot \neg M \sim x$
T1. $\sim Cx \equiv \neg Cx : ?Cx$
$\sim \neg Cx \equiv Cx : ?Cx$
$\sim ?Cx \equiv Cx : \neg Cx$
T2. $\neg Cx \vdash Mx$
T3. $Cx \vdash \neg Nx$, $\neg Cx \vdash Nx$
T4. $Nx \dashv\vdash \neg Cx$.

But $\neg Nx$ and Cx do not coincide; nor do $?Nx$ and $?Cx$.

The term "random" is used in various senses. In particular, an event is called random if it is the result of an unrepeatable coincidence, not the necessary consequence of given circumstances, etc. But, in such cases one can either do without it or, after a certain "purification", one can reduce it to the above-mentioned form of the term C.

10. MODALITY AND EXISTENCE

Let X be any of the sentences $(s^1,\ldots, s^n)\,\alpha \leftarrow P$ and $\sim((s^1,\ldots, s^n)\,\alpha \leftarrow P)$, where $n \geqslant 1$, and s^i and s^k are any of $s^1,\ldots, s^n$. The sentences "An s^i such that X is possible (necessary, impossible, etc.)", "There is a possible s^i and a necessary s^k such that X", etc., will be represented as

$$(Mod\, s^i)\, X,\ (Mod^1 s^i)\,(Mod^2 s^k)\, X,\ (Mod^1 s^1) \ldots (Mod^n s^n)\, X.$$

The properties of these sentences are defined by

$A1.\ (Mod\, s^i)\, X \equiv Mod \downarrow (s^i \downarrow PX \leftarrow E)$

$A2.\ (Mod^1\, s^i)\,(Mod^2\, s^k)\, X \equiv (Mod^1\, s^i)\, X \cdot (Mod^2\, s^k)\, X$

$(Mod^1\, s^1) \ldots (Mod^k s^k)\, X \equiv (Mod^1\, s^1)\, X \cdot \ldots \cdot (Mod^k s^k)\, X$

$(Mod^1\, s^1) \ldots (Mod^n s^n)\, X \equiv (Mod^1\, s^1)\, X \cdot \ldots \cdot (Mod^n s^n)\, X$

$A3.\ (Mod(s^i, s^k))\, X \equiv (Mod\, s^i)\,(Mod\, s^k)\, X$

$(Mod(s^1, \ldots, s^k))\, X \equiv (Mod\, s^1) \ldots (Mod\, s^k)\, X$

$(Mod(s^1, \ldots, s^n))\, X \equiv (Mod\, s^1) \ldots (Mod\, s^n)\, X.$

It is obvious that

$T1.\ (Mod^1\, s^i)\,(Mod^2\, s^k)\, X \dashv\vdash (Mod^2\, s^k)\,(Mod^1\, s^i)\, X$

$T2.\ (Mod(s^i, s^k))\, X \dashv\vdash (Mod(s^k, s^i))\, X.$

According to the definitions accepted we obtain sentences similar to sentences with quantifiers. But if for quantifiers we assume any combinations of $\alpha\rtimes^1$ and $\alpha\rtimes^2$ in $(\alpha\rtimes^1 t^1)\ (\beta\rtimes^2 t^2)\ X$, we cannot put some combinations of Mod^1 and Mod^2 for modal signs. This exclusion is defined by:

$A4.\ (Mod\, s^i)\, X \vdash (Mod\, s^k)\, X.$

Consequences of *A1–A4*:

$T3.\ \sim((Ms^i)\,(\neg\, Ms^k)\, X);$

similarly for the pairs N and $\neg\, N$, N and $\neg\, M$, M and $?M$, N and $?N$, N and $?M$. Not excluded are only those combinations of Mod^1 and Mod^2 for which any one of the assertions

$$(Mod^1\, s^i)\, X \vdash (Mod^2\, s^i)\, X,\ (Mod^2\, s^i)\, X \vdash (Mod^1\, s^i)\, X,$$
$$(Mod^1\, s^k)\, X \vdash (Mod^2\, s^k)\, X,\ (Mod^2\, s^k)\, X \vdash (Mod^1\, s^k)\, X.$$

holds.

$A5.\ (Ms)\, X\, (Ns \downarrow X)\, Y \vdash (Ms)\, Y$

$A6.\ (Ns)\, X\, (Ns \downarrow X)\, Y \vdash (Ns)\, Y$

11. MODALITY OF A HIGHER ORDER

We widen the notion of event by adopting the following definitions:

1) if X is a sentence, then x is the term of an event;

2) if x is the term of an event, then $x \downarrow Mod$ is the term of an event, where Mod is any of αN and αM;

3) if x is the term of an event, then $Mod\, x$ is the term of an event.

The terms indicated in points 2 and 3 are not identical in meaning.

$A1.\ Mod(x \downarrow Mod).$

holds for the first.

Consequences from $A1$:

$T1.\ \sim (\alpha M(x \downarrow \beta M))$
$T2.\ \sim (\alpha N(x \downarrow \beta N)),$

where α and β are different.

$A2.\ MMx \vdash Mx$
$A3.\ Nx \vdash NNx.$

hold for the others.

Consequences of $A2$ and $A3$:

$T3.\ MNx \vdash Mx$
$T4.\ NMx \vdash Mx.$

12. MODALITY AND QUANTIFIERS

There is a similarity between quantifiers and modal signs: to the assertions

$$Nx \equiv \neg M \sim x,\ Nx \vdash X,$$

etc., correspond the assertions

$$(\forall s)\, X \equiv (\neg \exists s) \sim X, \quad (\forall s)\, X \vdash X,$$

etc. There is a connection, for example: if s is free in X, then

$$(\exists s)\, X \vdash Mx.$$

In a series of cases they are used together with the sentences

$$(\alpha >\!\!| s^1)(Mod\, s^2)\, X \quad \text{and} \quad Mod(\alpha >\!\!| s)\, X.$$

The assertions

$A1.\ (\alpha >\!\!| s^1)(Mod\, s^2)\, X \equiv (\alpha >\!\!| s^1)((Mod\, s^2)\, X)$
$A2.\ Mod \downarrow ((\alpha >\!\!| s)\, X) \equiv (\alpha >\!\!| s)(Mod\, x).$
$A3.\ (Ms)\, X (\forall s \downarrow X)\, Y \vdash (Ms)\, Y$
$A4.\ (Ns)\, X (\forall s \downarrow X)\, Y \vdash (Ns)\, Y$

hold for their joint use.

13. MODALITY AND ENTAILMENT

There is a tradition (beginning with Lewis) of connecting modality with the notion of logical entailment. Entailment is defined through modal concepts or vice versa. Of course, by means of the notion of entailment one can define logical modality and show how to obtain in certain cases modal sentences in general. But this does not at all mean that modal concepts are defined through the concept of entailment. The definition of entailment through modal concepts is generally defective. There is no need thoroughly to develop here an argument in favor of our view. It is clear from the above considerations. We will add only the following.

Examining all possible relations of sentences with *Mod* and $\vdash$ (and also with the sign of weakened entailment), we find only one case with a positive result:

$$M(x\cdot\sim y)\rightarrow\sim(X\vdash Y),\ (X\vdash Y)\rightarrow\neg M(x\cdot\sim y)$$

In the rest of the cases the result is written in the negative assertions:

$$\sim((X\vdash Y)\rightarrow(Mx\cdot y)),\ \sim(\mathrm{N}(x\cdot y)\rightarrow(X\vdash Y)),$$

etc. Even in the case which serves as base for the definition of entailment we get a negative result. Namely: "If $\neg M(x\cdot\sim y)$ then $X\vdash Y$ does not follow from it". In fact, $\neg M(x\cdot\sim x\cdot\sim y)$; but $\sim(X\cdot\sim X\vdash Y)$ and $\sim(Y\vdash\sim(X\cdot\sim X))$. Definition of the logical entailment of Y from X as $\neg Mx\cdot\sim y$ means acceptance of statements of the type "Y logically follows from $X\cdot\sim X$" and "$\sim(X\cdot\sim X)$ logically follows from Y", etc., i.e., "paradoxes of strict implication", since $\neg M(x\cdot\sim x)$ and $N\sim(x\cdot\sim x)$.

The strong implication of Lewis is just a shorthand for $LN(x\supset y)$ and nothing more.

14. MODALITY AND CONDITIONALITY

The assertion

$$\neg M(x\sim y)\rightarrow(X\rightarrow Y)$$

is not true for the same reason as

$$\neg M(x\sim y)\rightarrow(X\vdash Y).$$

A1. $(X\rightarrow Y)\rightarrow\neg M(x\sim y)$

$A2.\ M(x \sim y) \rightarrow \neg(X \rightarrow Y)$

$A3.\ (X \rightarrow Y)\, Mx \vdash My$

$\quad\ (X \rightarrow Y)\, Nx \vdash Ny$

$A4.\ (X \rightarrow Y) \neg My \vdash \neg Mx \quad (X \rightarrow Y)?My \vdash \sim Mx$

$\quad\ (X \rightarrow Y) \neg Ny \vdash \neg Nx \quad (X \rightarrow Y)?Ny \vdash \sim Nx$

$A5.\ MxMy \neg(X \rightarrow \sim Y) \vdash M(xy).$

15. LINGUISTIC TRANSFORMATIONS

There are linguistic transformations of sentences, definable by

$A1.\ s\, Mod\, \alpha \leftarrow P \equiv Mod\,(s\alpha \leftarrow P).$

An example for $sMod\alpha \leftarrow P$: "The student can not pass the exam" (which is identical in meaning to "It is possible that the student not pass the exam").

16. TERMS

There are predicates, the meaning of which is defined (implicitly or explicitly) by assertions like

$$s\alpha \leftarrow Q \equiv Mod \downarrow (s\alpha \leftarrow P)$$
$$s^i\alpha \leftarrow Q \equiv Mod \downarrow ((s^1, \ldots, s^n)\, \alpha \leftarrow P)$$
$$(s^i, \ldots, s^k)\, \alpha \leftarrow Q \equiv Mod \downarrow ((s^1, \ldots, s^n)\, \alpha \leftarrow P).$$

Examples of such predicates: "soluble in water", "heat-conductive", "opaque", etc.

Of special interest here are predicates which fix potential (i.e., cases where *Mod* is M and α is empty). For example, let the P^1 and P^2 be such that

$$\sim ((s \leftarrow P^1) \cdot (s \leftarrow P^2)) \cdot M \downarrow (s \leftarrow P^1) \cdot M \downarrow (s \leftarrow P^2).$$

We adopt the definitions:

$$s \leftarrow Q^1 = Df.\, M \downarrow (s \leftarrow P^1) \quad \text{and} \quad s \leftarrow Q^2 = Df.\, M \downarrow (s \leftarrow P^2).$$

According to the condition and the definitions,

$$(s \rightarrow Q^1) \cdot (s \leftarrow Q^2)$$

will be true. For example, one and the same body cannot be heated to plus 20 °C and simultaneously cooled to minus 20 °C; but there are bodies which can be so heated and can be so cooled. Sometimes potential predicates of this type make logical contradictions seem valid.

$A1.\ s\, Mod\, \alpha \leftarrow P \equiv Mod \downarrow (s\alpha \leftarrow P)$

17. TRUTH-VALUES

Let *Mod* be any of *M* and *N*.

D1. $[\alpha\, Mod x] \leftarrow v^1$ if and only if $\alpha\, Mod x$, in fact. And this "in fact" can mean the correspondence of $\alpha\, Mod$ to the base and criteria of prognosis of a given science, or to the modal classification of events in a science, etc.

$$\begin{aligned}
&D2.\ [Mod\, x] \leftarrow v^4 \equiv [\neg\, Mod\, x] \leftarrow v^1 \\
&[\neg\, Mod\, x] \leftarrow v^4 \equiv [Mod\, x] \leftarrow v^1 \\
&[?\, Mod\, x] \leftarrow v^4 \equiv ([Mod\, x] \leftarrow v^1) : ([\neg\, Mod\, x] \leftarrow v^1) \\
&D3.\ [Mod\, x] \leftarrow v^2 \equiv [?\, Mod\, x] \leftarrow v^1 \\
&[\neg\, Mod\, x] \leftarrow v^2 \equiv [?\, Mod\, x] \leftarrow v^1 \\
&D4.\ [\alpha\, Mod\, x] \leftarrow v^3 \equiv [\sim M \downarrow (s \leftarrow E)] \leftarrow v^1 .
\end{aligned}$$

Truth-values for the other cases are established in accordance with the previously accepted assertions and definitions. For example,

$$[N(x \cdot y)] \leftarrow v^1 \dashv\vdash ([Nx] \leftarrow v^1) \cdot ([Ny] \leftarrow v^1).$$

Sentences of the type $\alpha\, Mod x$ cannot be presented as truth-functions of X. In the first place, in two-valued logic there is no such one-argument function. In many-valued (three or more values) logic it is possible to have one-argument functions of X, the interpretation of which as $\alpha\, Mod x$ is useful as a heuristic method for the investigation of some properties of modal sentences (e.g., for the definition of the class of tautologies). But this is not a description of the ways of establishing the truth-values of modal sentences.

For example, we take the sentence $M\downarrow(s \leftarrow P)$. If $[s \leftarrow P] \leftarrow v^1$, then $[M\downarrow(s \leftarrow P)] \leftarrow v^1$. But this happens not because of the definition of $[M(s \leftarrow P)] \leftarrow v^1$ but because of the assertion $X \vdash Mx$, where M does not occur in X. In all other cases the truth-value of $M\downarrow(s \leftarrow P)$ remains unknown: knowing only that $[s \leftarrow P] \rightarrow v^2$, v^3, or v^4, we cannot yet say anything about the truth-value of $M\downarrow(s \leftarrow P)$. The most varied combinations are possible here: it is possible that $[s \leftarrow P] \leftarrow v^4$, and $[M\downarrow s \leftarrow P)] \leftarrow v^2$; $[s \leftarrow P] \leftarrow v^4$ and $[M\downarrow(s \leftarrow P)] \leftarrow v^1$; it is possible that $[s \leftarrow P] \leftarrow v^4$ and $[M\downarrow(s \leftarrow P)] \leftarrow v^4$, etc. Similarly for N. There is only one case where we can decide the truth-value of Nx on the basis of that of X and even this has a purely negative form:

$$([X] \urcorner \leftarrow v^1) \rightarrow ([Nx] \urcorner \leftarrow v^1).$$

18. PROBABILITY

Possibilities of events are differentiated:

1) topologically (one event is more or less possible than another or as possible as another);

2) according to magnitude.

D1. The magnitude (degree) of possibility of an event is called the probability of the event. The probability of x will be written as

$$p(x).$$

One usually uses the numbers from 0 to 1 to represent probability. Probability is assigned to events according to definite rules which are the object of special study in the mathematical theory of probability. Within the limits of logic the properties of probability can be defined by

A1. $(p(x) = 0) \equiv \urcorner Mx$
A2. $(p(x) = 1) \equiv \urcorner M \sim x$
A3. $(0 < p(x)) \equiv Mx$
A4. $(p(x) < 1) \equiv M \sim x$
A5. $p(\sim x) = 1 - p(x)$
A6. $p(x^1 \cdot x^2 \cdot \ldots \cdot x^n) \leqslant min(p(x^1), p(x^2), \ldots, p(x^n))$
A7. $p(x:y) \geqslant max(min(p(x), 1 - p(y)), min(p(y), 1 - p(x)))$
$p(x^1:x^2:\ldots:x^n) \geqslant max(min(p(x^1), 1 - p(x^2), \ldots,$
$1 - p(x^n)), \ldots, \quad min(p(x^n), \quad 1 - p(x^1), \ldots, 1 - p(x^{n-1}))).$

Logical-philosophical works often talk about different concepts of probability. This terminology is not wholly accurate. There is one concept of probability. There are different ways of determining probability. The frequency method and equal probability method are well known.

In cases of prognoses relative to the occurrence of individual events one uses the method of "weighting" circumstances which are favorable or unfavorable to occurrence of the event. Here a number can be assigned to each relevant circumstance and then from the relations between them one calculates the probability of the event which interests the investigator.

For example, let the number α be the sum of all numbers assigned to all circumstances "pro" (the "weight" of these circumstances) and β the sum of numbers assigned to circumstances "con". Then the probability of the event will be $\alpha/(\alpha+\beta)$. This is usually done implicitly with the less accurate estimates: "bigger", "smaller", "much larger", etc.

The probability of a sentence is the probability of its truth. The expression "degree of truth of the sentence" is equivalent to the expression "degree of probability that the sentence is true" (i.e., it is equivalent to the expression "degree of probability of the event referred to in the given expression"). And there is no probability logic other than mathematical probability theory.

19. THE ACTUAL AND THE POTENTIAL

In various kinds of definitions of logical concepts one can find $\exists$ and M. Depending on which of these signs is found in the definition one gets definitions with the expressions "existentially" or "actually" (for $\exists$) or "potentially" (for M). These definitions are not equivalent, since

$$\sim(Mx \vdash (\exists s)\, X), \quad \sim((\neg\,\exists s)\, X \vdash \neg\, Mx)$$

(where s is free in X).

Let K^0t be the term "A finite subclass of Kt". We adopt the definitions:

D1. Kt is actually infinite if

$$(\forall K^0 t)\,(\exists t^i)\,(\sim(t^i \in K^0 t)\cdot(t^i \in Kt)),$$

and potentially infinite if

$$(\forall K^0 t)\,(M t^i)\,(\sim(t^i \in K^0 t)\cdot(t^i \in Kt)).$$

D2. Kt is actually finite if

$$(\neg\,\forall K^0 t)\,(\exists t^i)\,(\sim(t^i \in K^0 t)\cdot(t^i \in Kt)),$$

and potentially finite if

$$(\neg\,\forall K^0 t)\,(M t^i)\,(\sim(t^i \in K^0 t)\cdot(t^i \in Kt)).$$

20. BASIC MODAL LOGIC

Alphabet: M, N, C are modal predicates.

D1. If Q is a modal predicate, a is a subject, and X is a K-formula, then $\alpha Q \downarrow X$ and $(\alpha Q a)X$ are elementary modal formulae. The latter are K-formulae. A modal formula is a K-formula which contains an elementary modal formula.

In what follows the $\downarrow$ before K-formulae will disappear and we will use $x, y, \ldots$ for K-formulae

Axiomatic schemata *AI*:

1. $Nx \vdash X$
2. $X \vdash Mx$

Axiomatic schemata *AII*:

1. $Nx \dashv\vdash \neg M \sim x$
2. $\neg Nx \dashv\vdash M \sim x$

Axiomatic schemata *AIII*:

1. $M(x \vee y) \dashv\vdash Mx \vee My$
2. $\neg M(x \vee y) \dashv\vdash \neg Mx \neg My$

Axiomatic schemata *AIV*:

1. $M(xy) \vdash Mx$
2. $NxMy \vdash M(xy)$
3. $N(x \vee y) \vdash Nx \vee My$
4. $\neg Mx \vee \neg My \vdash \neg M(xy)$
5. $Mx?My \vdash ?M(xy)$
6. $?Mx?My \vdash ?M(xy)$

Axiomatic schemata *AV*:

1. $Cx \dashv\vdash MxM \sim x$
2. $\neg Cx \dashv\vdash Mx \neg M \sim x$

Axiomatic schemata *AVI*:

1. $\alpha Q(x^1, \ldots, x^n) \dashv\vdash \alpha Q x^1 \cdot \ldots \cdot \alpha Q x^n$
2. $\alpha Q(\cdot x^1, \ldots, x^n) \dashv\vdash \alpha Q x^1 \cdot \ldots \cdot \alpha Q x^n$
3. $\alpha Q(\vee\, x^1, \ldots, x^n) \dashv\vdash \alpha Q x^1 \vee \ldots \vee \alpha Q x^n$,

where Q is M, N or C.

Rule of inference:

R1. If $\vdash X$, then $\vdash Nx$.

Interpretation:

1) if x can (not) take the value v^1, then Mx takes the value v^1 (nv^1); if Mx takes the value v^1 (nv^1), then x can (not) take the value v^1;

2) Nx is equivalent to $\neg M{\sim}x$, $\neg Nx$ to $M{\sim}x$, αCx to $Mx\alpha M{\sim}x$.

T1. All the provable formulae in this system are tautologies.

21. NORMATIVE SENTENCES

Normative sentences are sentences on the allowing and requiring of activities; also their negations and indeterminate forms. The subjects of normative sentences are names of activities, the predicates are the normative expressions "allowed", "required", etc. The truth-value of normative sentences is established according to a general rule. For example, the sentence "Smoking forbidden" is true if smoking is really forbidden (if there is such a norm), false if smoking is not forbidden, and indeterminate if there is no prescription.

All normative sentences are local since they are sentences not about laws of nature but about norms established for certain people and by certain people. For example, "Smoking forbidden" is true in places where smoking is forbidden and false in places specially designated for smoking.

The truth-value of normative sentences does not depend on whether or not the activities they talk about are carried out. Everyone knows of cases where people do forbidden actions and avoid required ones, etc.

The establishment of norms has practical meaning in reference to subjectively free actions, i.e., relevant to possible and not necessary actions (for example, it is practically senseless to forbid, allow or demand the construction of perpetual motion machines).

The meaning of normative predicates is established according to a schema like the following. For every action *a* one can find an *X* or *Y* (or both) such that the assertions "If *a* is carried out, then *X*" and "If *a* is not carried out, then *Y*", where *X* and *Y* fix the consequences of the carrying out or non-carrying out of the action, hold. In many cases one does not consider the concrete results of the action: one simply presupposes some (bad or good, pleasant or unpleasant) consequences; one presupposes some punishment or reward.

Normative predicates have certain peculiarities from the point of view of theory of inference. Normative logic deals with these. In our system normative logic (the theory of normative sentences) is formed through the following additions to our earlier systems:

Designations:

1) $a, a^1, a^2, \ldots$ are action variables.

2) O, P, H are normative predicates (corresponding to "is obligatory", "is allowed", "is neutral" or "is indifferent").

D 1. Action terms:

1) action variables are action terms;

2) if $b, b^1, \ldots, b^n$ $(n \geqslant 2)$ are action terms, then $\tilde{b}$, $(b^1 \cdot \ldots \cdot b^n)$ and $(b^1 \vee \ldots \vee b^n)$ are action terms;

3) something is an action term only in function of 1 and 2.

D2. Action terms are subjects and normative predicates are predicates. If Q is a normative predicate and b is an action term, then $Q(b)$, $\neg Q(b)$ and $?Q(b)$ are K-formulae. A normative formula is a formula containing a formula of the type $\alpha Q(b)$.

Axiom *AI*:

$$O(a) \vdash P(a)$$

Axioms *AII*:

1. $O(a) \dashv\vdash \neg P(\tilde{a})$
2. $\neg O(a) \dashv\vdash P(\tilde{a})$
3. $?O(a) \dashv\vdash ?P(\tilde{a})$

Axioms *AIII*:

1. $P(a^1 \vee a^2) \dashv\vdash P(a^1) \vee P(a^2)$
2. $\neg P(a^1 \vee a^2) \dashv\vdash \neg P(a^1) \neg P(a^2)$
3. $P(a^1 a^2) \vdash P(a^1)\, P(a^2)$
4. $\neg P(a^1) \vee \neg P(a^2) \vdash \neg P(a^1 a^2)$
5. $O(a^1)\, P(a^2) \vdash P(a^1 a^2)$
6. $O(a^1 \vee a^2) \vdash O(a^1) \vee P(a^2)$
7. $P(a^1) ?P(a^2) \vdash ?P(a^1 a^2)$
8. $?P(a^1) ?P(a^2) \vdash ?P(a^1 a^2)$

Axioms *AIV*:

1. $H(a) \dashv\vdash P(a)\,P(\tilde{a})$
2. $\neg H(a) \dashv\vdash \neg P(a) \vee \neg P(\tilde{a})$
3. $?H(a) \dashv\vdash ?P(a) \vee ?P(\tilde{a})$

Rules of inference:

R1. Substitution of normative predicates for predicate variables and of action terms for subject variables.

R2. Substitution of action terms for action term variables.

R3. Substitution of normative formulae for sentential variables.

Interpretation:

1) if a can (not) take the value v^1, then $P(a)$ takes the value v^1 (nv^1); if $P(a)$ has the value v^1 (nv^1), then a can (not) take the value v^1;

2) the formulae found to the left and right of the sign $\dashv\vdash$ in axioms *AII* and *AIV* are equivalent.

T1. All provable normative formulae are tautologies (theorem of consistency).

D3. The prime normative formulae are $P(b)$ and $\neg P(b)$, where b is an action term variable.

D4. A reduced normative formula is obtained from a normative formula through replacement of all occurrences of normative formulae by such formulae as contain only prime normative formulae; the substitution takes place in accordance with *AII–AV* and the formula $?P(a) \dashv\vdash \sim P(a) \sim \neg P(a)$, which is provable in our system.

T2. If $X \vdash Y$ is provable then the reduced formula $X^* \vdash Y^*$, corresponding to it, is provable and vice versa.

T3. If $X \vdash Y$ is provable and $X^* \vdash Y^*$ is its reduced form then no prime normative formulae lacking in X^* occur in Y^* (theorem of non-paradoxicality).

It should be noted that the term (bc) is not read as "every one of the actions b and c" or as "an action, made up of actions b and c", but as "an action called b and also called c". Analogously for $(b \vee c)$. For the first of these cases, we have the axiomatic schema *AV*:

1. $\alpha Q(\cdot a^1, \ldots, a^n) \dashv\vdash \alpha Q(a^1) \cdot \ldots \cdot \alpha Q(a^n)$
2. $\alpha Q(\vee a^1, \ldots, a^n) \dashv\vdash \alpha Q(a^1) \vee \ldots \vee \alpha Q(a^n)$,

where Q is O, P or H.

22. MODALITIES AND NORMS

There is some analogy between modal and normative predicates. Both are logically interchangeable. There are closely similar assertions, like

$$\begin{array}{ll} Nx \vdash Mx & O(a) \vdash P(a) \\ N(xy) \vdash NxNy & O(ab) \vdash O(a)\,O(b), \end{array}$$

etc. But, there is an important difference between them. Provable assertions of the type $\vdash A$ which are not the result of substitution in a law of the general theory of deduction hold for modal sentences. For example, $\vdash N(x \vee {\sim} x)$, $\vdash \neg M(x {\sim} x)$, etc. There are no such laws for normative sentences. Here there are cases where $P(a)$ and $P(\tilde{a})$ are simultaneously true, i.e., $P(a \cdot \tilde{a})$ is not a contradiction; and there are cases where $\neg O(a)$ and $\neg O(\tilde{a})$ are simultaneously true, i.e., $O(a \vee \tilde{a})$ is not a law, etc.

Further, the happening or not happening of events makes it possible in many cases to determine the truth-values of modal sentences. Thus, if X is true, then Mx is true; if X is false, then Nx is false. There is nothing similar for normative sentences. Even if action a is carried out this does not mean that $P(a)$ is true (i.e., that the action a is allowed); if a is not carried out this does not mean that $O(a)$ is false (i.e., that the action a is not required).

CHAPTER SEVENTEEN

RELATIONS

1. SENTENCES ABOUT RELATIONS

Among the sentences with many-place predicates there are some which have no parts which are themselves sentences. They have the structure

$$s^1 R s^2 \quad \text{or} \quad (s^1, \ldots, s^n) R$$

or can be reduced to such a form through linguistic transformations which do not change their sense. For example, the sentences "*a* is bigger than *b* by two" can be transformed into "*a* is twice as big as *b*", where "twice as big as" is the *R*. These sentences and their intrinsic negations

$$s^1 \urcorner R s^2 \quad \text{and} \quad (s^1, \ldots, s^n) \urcorner R,$$

and the indeterminate forms

$$s^1 ?R s^2 \quad \text{and} \quad (s^1, \ldots, s^n) ?R$$

and their extrinsic negations are called sentences about relations or, for short, r-sentences. That which is talked about in such sentences are called relations. Different indices on R^1, R^2,... will indicate that the relations differ somehow ("is larger than", "is farther than", etc.).

We note that in r-sentences *R* is part of the sentence but not a predicate. For example, in the sentences "*a* is larger than *b*" the expression "is larger than" is *R* and the predicate *P* will have the form "the first is larger than the second".

R-sentences are special cases of the sentences examined above. For them, certain corresponding assertions hold, like:

T1. $\sim(s^1 \alpha R s^2) \equiv (s^1 \beta R s^2) \vee (s^1 \gamma R s^2)$, where αR, βR and γR differ like R, $\urcorner R$ and $?R$ in any distribution.

T2. $\sim((s^1 R s^2) \cdot (s^1 \urcorner R s^2))$, $\sim((s^1 R s^2) \cdot (s^1 ?R s^2))$,

$$\sim((s^1 \urcorner R s^2) \cdot (s^1 ?R s^2)).$$

2. LOGICAL TYPES OF RELATIONS

From a purely logical point of view relations are classified as follows:

D1. (R is reflexive)$\leftrightarrow(sRs)$;
(R is areflexive)$\leftrightarrow(s\urcorner Rs)$;
(R is non-reflexive)$\leftrightarrow\sim(sRs)$.

D2. (R is symmetrical)$\leftrightarrow(s^1Rs^2\rightarrow s^2Rs^1)$;
(R is asymmetrical)$\leftrightarrow(s^1Rs^2\rightarrow s^2\urcorner Rs^1)$;
(R is non-symmetrical)$\leftrightarrow(s^1Rs^2\rightarrow\ \sim(s^2Rs^1))$;
(R is weakly non-symmetrical)$\leftrightarrow\sim(s^1Rs^2\rightarrow s^2Rs^1)$.

D3. (R is transitive)$\leftrightarrow(s^1Rs^2\cdot s^2Rs^3\rightarrow s^1Rs^3)$;
(R is atransitive)$\leftrightarrow(s^1Rs^2\cdot s^2Rs^3\rightarrow s^1\urcorner Rs^3)$;
(R is non-transitive)$\leftrightarrow(s^1Rs^2\cdot s^2Rs^3\rightarrow\ \sim(s^1Rs^3))$;
(R is weakly non-transitive)$\leftrightarrow\sim(s^1Rs^2\cdot s^2Rs^3\rightarrow s^1Rs^3)$.

3. SIMPLE AND COMPLEX RELATIONS

Relations are simple or complex depending on the terms corresponding to them. Complex relational terms are formed according to the rules

$$s^1(R^1\cdot R^2)\,s^2\equiv(s^1R^1s^2)\cdot(s^1R^2s^2)$$
$$s^1(R^1\vee R^2)\,s^2\equiv(s^1R^1s^2)\vee(s^1R^2s^2),$$

etc., examined in Section 8 of Chapter Seven.

4. ELEMENTARY AND DERIVATIVE RELATIONS

D1. r-sentences of the type

$$s^1\alpha Rs^2,$$

where α indicates the presence or absence of $\urcorner$ or ?, are elementary r-sentences.

D2. If X is an r-sentence and r-sentences $Z^1,\ldots,Z^n$ $(n\geqslant 2)$ occur in Y and $X\equiv Y$, then X is an r-sentence derived from $Z^1,\ldots,Z^n$. For example, "$s^1,\ldots,s^n$ are equal to each other" is identical in meaning with "s^1 is equal to s^2",..., "s^{n-1} is equal to s^n"; $(s^1\cdot s^2)Rs^3\equiv s^1Rs^3\cdot s^2Rs^3$; "$s^1$ is the grandmother of s^2" is equivalent in meaning to "There exists an s^3 such that if s^1 is the mother of s^3, then s^3 is the father or mother of s^2".

We assume that the sentences

$$(s^1, \ldots, s^n)\,\alpha R$$

are derived from elementary ones. This is an empirically given fact. We shall therefore limit our attention to elementary r-sentences.

Elementary r-sentences either are presuppositions and stipulations or they fix direct observations, or they are obtained from sentences with one-place predicates (from one-subject sentences). For example, "*a* is twice as large as *b*" can be obtained from sentences about the magnitude of *a* and about that of *b* independently of each other. The situation in science is: if there is a need to reduce r-sentences to one-subject sentences then this is done one way or another. One can formulate the following principle: "For every r-sentence X one can find a set of at least two one-subject sentences $Y^1, \ldots, Y^m$ such that $X \equiv Y^1 \cdot \ldots \cdot Y^m$". The search for methods of such a reduction is one of the possible tasks of the science in which X is formulated.

5. BINARY AND *n*-ARY RELATIONS

From the structural point of view the simplest r-sentences are binary: they are sentences on binary (between two objects) relations. They have the form $s^1 \alpha R s^2$. But the terms s^1 and s^2 can have the form $(s^1_1, \ldots, s^1_n)$ and $(s^2_1, \ldots, s^2_m)$. In such a case we obtain *n*-ary r-sentences, where "*n*-" is greater than two: they are sentences about *n*-ary (ternary and more) relations. However, this is so only when the *n*-tuple of two and more terms is s^2 and not s^1.

This is because the sentences $s^1 \alpha R s^2$ are not just sentences about relations between objects; they are sentences about relations of some objects to others: 1) single object s^1 to single object s^2; 2) single object s^1 to *n*-tuple $(s^2_1, \ldots, s^2_m)$; 3) *n*-tuple $(s^1_1, \ldots, s^1_n)$ to a single object s^2; 4) *n*-tuple $(s^1_1, \ldots, s^1_n)$ to *n*-tuple $(s^2_1, \ldots, s^2_m)$. Thus the number of objects in a given relation does not depend on the form of term s^1. It depends on the form of s^2: if s^2 is one term, then the relation is binary: if s^2 is $(s^2_1, \ldots, s^2_m)$, then the relation is $(m+1)$-ary. We call object s^1 the first member of the relation and s^2 or $(s^2_1, \ldots, s^2_m)$ the second member.

There are cases where *n*-ary (where "*n*-" is greater than two) r-sentences are reduced to binary ones. For example, "*a* is found between *b* and *c*" can sometimes be shorthand for "*a* is farther than *b*, *c* is farther than *a*". But there are also cases when such a reduction is not effected. For example, for meaningful operation with the sentence "Bologoe is between Moscow and Leningrad" it is sometimes enough to know that if one takes the train from Moscow to Leningrad (or vice versa) one has to pass through Bologoe. From the logical point of view the ternary sentence is not reduced here to a set of binary ones.

The factual situation in science is: if it is necessary to reduce some *n*-ary r-sentence to binary ones then a solution is somehow found. Thus we can formulate the principle: "For every r-sentence X one can construct a non-empty set of binary r-sentences $Y^1, \ldots, Y^m$ such that $X \equiv Y^1 \cdot \ldots \cdot Y^m$. Z, where Z is a sentence or there is none (it lacks)". Discovery of the methods of such a reduction is one of the possible tasks of the science in which X is formulated.

6. UNIVERSAL AND LOCAL RELATIONS

D1. The relation $s^1 \alpha R s^2$ is actually universal if $(\exists\,(s^1, s^2))\,(s^1 \alpha R s^2)$ $(\neg\exists\,(s^1, s^2))\,(s^1 \beta R s^2)$, where α and β are different, and actually local if $(\exists\,(s^1, s^2))\,(s^1 \alpha R s^2)\,(\exists\,(s^1, s^2))\,(s^1 \beta R s^2)$. Definitions of potentially local and universal are obtained by putting M for $\exists$.

7. PSEUDORELATIONS

In logical-mathematical literature a function with two or more arguments is often called relation. For example, every two natural numbers can be put in correspondence with a natural number which is called their sum: and the sum of two numbers is considered their relation.

The general schema for the formation of terms in such cases has the following form:

1) we establish

$$t^k \Leftarrow f(t^1, \ldots, t^n);$$

2) we adopt

$$t(t^1, \ldots, t^n) = Df.\, t^k.$$

But terms of this type are not terms of relations in our sense. These are relations of the type

$$\downarrow(s^1 \alpha R s^2),$$

where all the terms s^1, R and s^2 are independent from each other in meaning. But the term $t(t^1,\ldots, t^n)$ clearly depends for meaning on $t^1,\ldots, t^n$. It literally means: "The function t of $t^1,\ldots, t^n$".

8. COMPARISON

Two objects s^1 and s^2 can be such that

$$(\exists P^*)((s^1 \leftarrow P^*)\cdot(s^2 \leftarrow P^*))$$
$$(\exists P^*)((s^1 \leftarrow P^*)\cdot(s^2 \neg\leftarrow P^*)).$$

In such cases one talks about the similarity and difference of objects. Depending on the number of attributes and on their importance one talks about lesser or greater similarity or difference. One constructs the r-sentences "s^1 is similar to s^2", "s^1 is not very similar to s^2", "s^1 is strongly different from s^2", etc.

Identity of objects is a special case of the comparison of objects (and the same is true of its opposite). Sentences on the identity of objects are constructed as follows:

1) in each case one establishes (explicitly or implicitly) a set of attributes $P^1,\ldots, P^n$ such that "If $(s^1 \leftarrow P^1\cdot\ldots\cdot P^n)\cdot(s^2 \leftarrow P^1\cdot\ldots\cdot P^2)$, then s^1 is identical to s^2";

2) depending on whether or not this condition is met, one accepts "s^1 is identical to s^2" or its negation (indeterminacy is excluded here).

The relation of identity is reflexive, symmetrical and transitive and its negation is areflexive, symmetrical and non-transitive.

Conceivable for abstract objects are cases where one assumes

$$(\forall P^*)((s^1 \leftarrow P^*)\cdot(s^2 \leftarrow P^*))$$
$$(\forall P^*)((s^1 \neg\leftarrow P^* \downarrow s^2)\cdot(s^2 \neg\leftarrow P^* \downarrow s^1)),$$

etc., (cases of absolute identity and difference). The limit case of the identity of objects is establishing that s^1 and s^2 are one and the same object. But here again one must indicate the attributes, the presence of which is considered sufficient for the establishment of an identity of this kind (i.e., for the establishment that $s^1 \rightleftarrows s^2$).

Finally, we note the relations of superiority of some objects over others ("better", "worse", "more convenient", "richer", "much more beautiful", "not very interesting", etc.), established through comparison of sets of attributes. Sentences of this type are constructed as follows:

1) one establishes that

$$s^1 \leftarrow P^1 \cdot \ldots \cdot P^n$$
$$s^2 \leftarrow P_1 \cdot \ldots \cdot P_m ;$$

2) from comparison of $P^1 \cdot \ldots \cdot P^n$ and $P_1 \cdot \ldots \cdot P_m$ is obtained a sentence on the superiority of one of s^1 and s^2 over the others (or its negation), depending on the character of these sets of attributes and on the accepted criteria of introduction of signs of superiority.

Relations of superiority are areflexive, asymmetrical and not always transitive. Thus, if s^1 is a little superior to s^2 and s^2 is a little superior to s^3, it does not follow that s^1 is a little superior to s^3. What is more, one here presupposes some context which gives the terms of these relations meaning and which allows them to serve certain purposes.

The comparison of objects has to be distinguished from comparison of the magnitudes of their attributes, the result of which is the sentences

$$(P \downarrow s^1)\, \alpha R (P \downarrow s^2),$$

where R can be the relation "equal", "larger", "much larger", "a times larger", etc. After linguistic transformation, these sentences assume the form

$$s^1 \alpha R P s^2 ,$$

where P is the name of the attribute, according to which the comparison is effected. For example, the sentence "s^1 is heavier than s^2" is obtained from the sentence "The weight of s^1 is greater than the weight of s^2".

The ability to evaluate attributes numerically is useful in the case of comparison of objects according to two and more different attributes. For example, let the magnitudes $\alpha^1, \ldots, \alpha^n$ be the numerical evaluations of the attributes of s^1 and $\beta^1, \ldots, \beta^2$ the same for s^2. For the comparison of s^1 and s^2 one need only calculate the average weight of the attributes of s^1 (say α^*) and that of the attributes of s^2 (say β^*). Now an r-sentence about objects s^1 and s^2 is obtained from the comparison of α^* and β^*.

Comparative sentences can all be reduced to sentences with one-place predicates in the following sense: if we have the comparative sentence Z with subjects s^1 and s^2, then in principle we can find sentences $s^1 \leftarrow PX$ and $s^2 \leftarrow PY$ such that "If $X \cdot Y$, then Z". But this does not mean that such a reduction is always effected in practice. Comparative sentences often fix directly observed facts (thanks to the existence of the corresponding habits).

The sentence that s^1 surpasses s^2 according to P will be written as

$$(s^1 > s^2 \mid P).$$

$(s^1 \neg > s^2 \mid P)$ and $(s^1 ? > s^2 \mid P)$ are, respectively, its partial negation and indeterminate form.

$$\begin{aligned}
&D1.\ (s^1 < s^2 \mid P) \equiv (s^2 > s^1 \mid P) \\
&(s^1 \neg < s^2 \mid P) \equiv (s^2 > s^1 \mid P) : (s^1 \neg > s^2 \mid P)(s^2 \neg > s^1 \mid P) \\
&(s^1 ? < s^2 \mid P) \equiv (s^2 ? > s^1 \mid P) \\
&D2.\ (s^1 = s^2 \mid P) \equiv (s^1 \neg > s^2 \mid P)(s^2 \neg > s^1 \mid P) \\
&(s^1 \neg = s^2 \mid P) \equiv (s^1 > s^2 \mid P) : (s^2 > s^1 \mid P) \\
&(s^1 ? = s^2 \mid P) \equiv ((s^1 ? > s^2 \mid P) \vee (s^2 ? > s^1 \mid P)) \sim \\
&\sim (s^1 > s^2 \mid P) \sim (s^2 > s^1 \mid P)
\end{aligned}$$

9. RELATIONS OF ORDER

We take the expression "the order of objects" to be primitively clear, subject only to examples and explanations. The distribution of objects in space and their coming-to-be and passing away in time are particular cases of order. To fix them we use the expressions "first", "second",..., "higher", "lower", "earlier", "simultaneous", "to the right of", etc.

Ordinal relations are a special case of relations of superiority. But it is a special case which is basic to many other relations and which has great importance in science.

For some objects the order is considered given (it is clear; it leaves no room for doubt), for others it is established through the first. The investigator determines the order of the objects either relative to itself or relative to another object selected for this purpose. Thus, in affirming that "*a* is farther than *b*" the speaker can have in mind that if some object is moved away from him in the direction of *a* it will pass *b*; and affirming that "Moscow is more to the south than Leningrad" one can have in mind that the latitude of Moscow is closer to the Equator than that of

Leningrad. If the investigator defines the order of two objects relative to a third which is different from the investigator himself then all its signs have a meaning as if the investigator were found in place of the object.

Fixing of the order of objects relative to the investigator is unreliable and subjective. In the other case one can choose convenient objects different from the investigator (we call them points of definition or of reference): these are constant, recognized and standard; they make it possible to verify the appropriate sentences and they exclude ambiguity, etc.

The sentence that s^1 precedes in order s^2 relative to s (s is the point of reference of the order of s^1 and s^2) will be expressed as

$$(s^1 > s^2 \mid s).$$

$(s^1 \neg > s^2 \mid s)$ and $(s^1 ? > s^2 \mid s)$ are, respectively, its partial negation and indeterminate form. The sign $>$ can be read as "is earlier than", "is to the right of", "is to the left of", "is later than", etc. The definitions of the signs $<$ (*D1*) and $=$ (*D2*) are analogous to *D1* and *D2* of the previous section. The point of reference of an order is often not given since it is assumed to be known. The ordering of objects is often confused with the establishment of relations of superiority in reference to some attribute (e.g., military rank). Where the order of objects is fixed by the objects themselves their order is their attribute. Otherwise the ordering of objects is done on the basis of comparing some of their attributes. Here the method of ordering (comparison) is difficultly distinguishable from its result – the fact of order.

10. THE LOGIC OF RELATIONS

Alphabet:

1) $>$, $<$, $=$ are two-place predicate constants (respectively, "precedes", "follows", "equals");

2) R, R^1, R^2,... are two-place predicate variables of order.

D1. If Q is a predicate, indicated in points 1 and 2, and a, b and c are subject variables, then $Q(a, b) \mid c$ is an elementary formula of the logic of relations. An elementary formula of the logic of relations is a K-formula. A formula in which an elementary formula of the logic of relations occurs is a formula of the logic of relations.

In what follows we will use $a\alpha Qb \mid c$ for $\alpha\, Q(a, b) \mid c$ as more convenient.

Axioms of comparison *AI* (for simplicity we will drop the parts of formulae having the form $\mid \downarrow c$):

1. $\vdash(s \neg > s)$
2. $(s^1 > s^2)\vdash(s^2 \neg > s^1)$
3. $(s^1 ? > s^2)\vdash(s^2 ? > s^1)$
4. $(s^1 \neg > s^2)\vdash(s^2 > s^1):(s^1 = s^2)$
5. $(s^1 > s^2)(s^2 > s^3)\vdash(s^1 > s^3)$
6. $(s^1 > s^2)(s^2 = s^3)\vdash(s^1 > s^3)$
7. $(s^1 \neg > s^2)(s^2 \neg > s^3)\vdash(s^1 \neg > s^3)$
8. $(s^1 ? > s^2)(s^2 = s^3)\vdash(s^1 ? > s^3)$
9. $(s^1 < s^2)\dashv\vdash(s^2 > s^1)$
10. $(s^1 \neg < s^2)\dashv\vdash(s^1 > s^2):(s^1 = s^2)$
11. $(s^1 ? < s^2)\dashv\vdash(s^2 ? > s^1)$
12. $(s^1 = s^2)\dashv\vdash(s^1 \neg > s^2)(s^2 \neg > s^1)$
13. $(s^1 \neg = s^2)\dashv\vdash(s^1 > s^2):(s^2 > s^1)$
14. $(s^1 ? = s^2)\dashv\vdash(s^1 ? > s)(s^2 \neg > s^1):(s^2 ? > s^1)(s^1 \neg > s^2):$
    $:(s^1 ? > s^2)(s^2 ? > s^1)$

Axioms of comparison *AII*:

1. $P(s^1) \neg P(s^2)\vdash(s^1 > s^2 \mid P)$
2. $\neg P(s^1) \neg P(s^2)\vdash(s^1 = s^2 \mid P)$
   $?P(s^1)?P(s^2)\vdash(s^1 = s^2 \mid P)$
3. $P(s^1)?P(s^2)\vdash(s^1 ? > s^2 \mid P)$
   $?P(s^1) \neg P(s^2)\vdash(s^1 ? > s^2 \mid P)$

Axioms of order *AIII* are obtained from *AI* by putting *s* for *P* throughout.

T1. $(s^1 = s^2 \mid \delta)\vdash(s^2 = s^1 \mid \delta)$.

T2. The relation > is areflexive, asymmetrical and transitive.

T3. $\sim(s > s \mid \delta)$
T4. $(s^1 > s^2 \mid \delta)\vdash \sim(s^2 > s^1 \mid \delta)$
T5. $\sim((s^1 > s^2 \mid \delta)\cdot(s^2 > s^1 \mid \delta))$
T6. $\sim((s^1 > s^2 \mid \delta)\cdot(s^1 \neg > s^2 \mid \delta))$.

T7. If δ^1 and δ^2 are different, then

$$\sim((s^1\alpha > s^2 \mid \delta^1) \vdash (s^1\alpha > s^2 \mid \delta^2))$$
$$\sim((s^1\alpha > s^2 \mid \delta^1) \vdash (s^1\alpha < s^2 \mid \delta^2)),$$

etc., for all other combinations of ordinal relations to the left and right of the implication sign.

11. THE RELATION "BETWEEN"

We designate the relation "between" with M and define it as follows:

A1. $(s^3\alpha\,\mathrm{M}(s^1, s^2)) \equiv (\alpha\exists\delta)((s^1 > s^3 \mid \delta)\cdot(s^3 > s^2 \mid \delta))$
$(s^4\alpha\,\mathrm{M}(s^1, s^2, s^3)) \equiv (\alpha\exists s^5)((s^5\,\mathrm{M}(s^1 s^2))\cdot(s^4\,\mathrm{M}(s^3, s^5)))$

. . . .

$(s^n\alpha\,\mathrm{M}(s^1, \dots, s^n)) \equiv (\alpha\exists s^{n+1})((s^{n+1}\,\mathrm{M}(s^1, \dots, s^{n-1}))\cdot$
$\cdot(s^k\,\mathrm{M}(s^n, s^{n+1})))$

Consequences of *A1*:

T1. $(s^3\,\mathrm{M}(s^1, s^2))\cdot(s^4\,\mathrm{M}(s^3, s^1)) \vdash (s^4\,\mathrm{M}(s^1, s^2))$
T2. $(s\,\mathrm{M}(s^1, \dots, s^n))\cdot(s^1\,\mathrm{M}(s_1, \dots, s_m))\cdot\ldots\cdot(s^n\,\mathrm{M}(s_1, \dots, s_m)) \vdash$
$\vdash (s\,\mathrm{M}(s_1, \dots, s_m))$
T3. $(s_1\,\mathrm{M}(s^1, \dots, s^{n+1}))\cdot(s_2\,\mathrm{M}(s_1, s^1 \dots s^n)) \vdash (s_2\,\mathrm{M}(s^1, \dots, s^{n+1}))$
T4. $s^3\alpha\,\mathrm{M}(s^1, s^2) \vdash s^3\alpha\,\mathrm{M}(s^2, s^1)$.

12. INTERVAL

Any two objects of a given domain of investigation, for which it is possible to establish an ordinal relation, can be related by a sign called the sign of the interval between s^1 and s^2. We will designate it by

$$\langle s^1, s^2 \mid \delta\rangle,$$

where δ is the point of reference. In other words:

D1. If $(s^1 > s^2 \mid \delta)$ or $(s^1 = s^2 \mid \delta)$, then $\langle s^1, s^2 \mid \delta\rangle$ is the sign of the interval between s^1 and s^2. These latter are the limits of the interval.

One can talk about the order of intervals, having in mind the ordinal relations of their limits. Different combinations are possible here: for example, let α^1 be the interval $\langle s^1, s^2 \mid \delta\rangle$, and α^2 the interval $\langle s^3, s^4 \mid \delta\rangle$. We adopt the definition:

D2. If $(s^1 > s^2 \mid \delta)$, then $(\alpha^1 > \alpha^2 \mid \delta) \equiv (s^2 > s^3 \mid \delta) : (s^2 = s^3 \mid \delta)$; if $(s^1 = s^2 \mid \delta)$, then $(\alpha^1 > \alpha^2 \mid \delta) \equiv (s^2 > s^3 \mid \delta)$.

13. ORDERED SERIES

D1. We will say that the elements of *Ks* form an ordered series

$$R(s, \delta),$$

if and only if

$$(\forall (s^{*1} \downarrow \in Ks, s^{*2} \downarrow \in Ks))((s^{*1} > s^{*2} \mid \delta) : (s^{*2} > s^{*1} \mid \delta)).$$

The elements of such a *Ks* are elements of $R(s, \delta)$.

T1. If s^1 and s^2 are elements of $R(s, \delta)$, then

$$\sim (s^1 > s^2 \mid \delta) \equiv (s^2 > s^1 \mid \delta).$$

D2. The series $R(s^1, \delta)$ is an actual segment of the series $R(s^2, \delta)$ if and only if

$$(\forall s^{*1} \downarrow \in Ks^1)(\exists s^{*2} \downarrow \in Ks^2)(\exists s^{*3} \downarrow \in Ks^2)(((s^{*2} < s^{*1} \mid \delta) :$$
$$: (s^{*2} = s^{*1} \mid \delta)) \vee ((s^{*3} > s^{*1} \mid \delta) : (s^{*3} = s^{*1} \mid \delta)));$$

the series $R(s^1, \delta)$ is a proper actual segment of series $R(s^2, \delta)$ if and only if

$$(\forall s^{*1} \downarrow \in Ks^1)(\exists s^{*2} \downarrow \neg \in Ks^1 \cdot \in Ks^2)(s^{*2} > s^{*1} \mid \delta) \vee$$
$$\vee (\forall s^{*1} \downarrow \in Ks^1)(\exists s^{*3} \downarrow \neg \in Ks^1 \cdot \in Ks^2)(s^{*3} < s^{*1} \mid \delta);$$

definitions of potential segment and proper potential segment are obtained by substituting *M* for both $\exists$.

D3. The series $R(s, \delta)$ is actually discontinuous if and only if

$$(\exists (s^{*1} \downarrow \in Ks, s^{*2} \downarrow \in Ks))(\neg \exists s^{*3} \downarrow \in Ks)((s^{*1} > s^{*3} \mid \delta) \cdot$$
$$\cdot (s^{*3} > s^{*2} \mid \delta));$$

the definition of potential discontinuity is obtained by putting $\neg M$ for $\neg \exists$ (the first $\exists$ remains).

D4. The interval $\langle s^{*1}, s^{*2} \mid \delta \rangle$ in *D3* is an interruption of $R(s, \delta)$.

D5. The series $R(s, \delta)$ is actually continuous if and only if

$$(\forall (s^{*1} \downarrow \in Ks, s^{*2} \downarrow \in Ks))(\exists s^{*3} \downarrow \in Ks)((s^{*1} > s^{*3} \mid \delta) \cdot$$
$$\cdot (s^{*3} > s^{*2} \mid \delta));$$

the definition of potential continuity is obtained by putting M for $\exists$.

D5 can be obtained from *D3* (and vice versa) if we consider one as the extrinsic negation of the other: *T1* holds and indeterminacy of the quantifiers is excluded. Series can be mixed, i.e., have continuous and discontinuous segments in any combination.

D6. s^i is the initial element of $R(s, \delta)$ if and only if

$$s^i \leftarrow P((\forall s^* \downarrow \in Ks)((s^* \downarrow \in Ks > s^i \mid \delta) : (s^* \downarrow \in Ks = s^i \mid \delta))),$$

and the final element of it if and only if

$$s^i \leftarrow P((\forall s^* \downarrow \in Ks)((s^* \downarrow \in Ks < s^i \mid \delta) : (s^* \downarrow \in Ks = s^i \mid \delta))).$$

D7. The series $R(s, \delta)$ has an actual initial element if

$$(\exists s^{*1} \downarrow \in Ks)(\forall s^{*2} \downarrow \in Ks)(s^{*1} > s^{*2} \mid \delta),$$

and does not have it if

$$(\neg \exists s^{*1} \downarrow \in Ks)(\forall s^{*2} \downarrow \in Ks)(s^{*1} > s^{*2} \mid \delta);$$

definitions for a potential initial element are obtained by putting M for $\exists$ and $\neg M$ for $\neg \exists$.

D8. Definitions of the final element of $R(s, \delta)$ are obtained from *D7* by putting $s^{*1} < s^{*2}$ for $s^{*1} > s^{*2}$.

Series are divided into those with a finite number of elements and those with an infinite number (actually and potentially). Dependence pertains here, too. For example, if a series is continuous or has continuous segments, or does not have an initial or final element, or has segments without initial or final elements, then it is infinite as to number of elements.

D9. For every series one can select another series such that the result of their comparison is a sign which is called the sign of the direction of the first series.

14. THE LENGTH OF AN INTERVAL AND OF A SERIES

Every interval has a length (or magnitude) which is characterized by the following assertions:

A1. $(s^1 = s^2 \mid \delta) \rightarrow \langle s^1, s^2 \mid \delta \rangle = 0$.

A2. $(s^1 > s^2 \mid \delta) \rightarrow \langle s^1, s^2 \mid \delta \rangle > 0$.

A3. $\langle s^1, s^2 \mid \delta \rangle = \langle s^2, s^1 \mid \delta \rangle$.

A4. $(s^1 > s^2 \mid \delta)\cdot(s^2 > s^3 \mid \delta) \rightarrow (\langle s^1, s^3 \mid \delta \rangle = \langle s^1, s^2 \mid \delta \rangle + \langle s^2, s^3 \mid \delta \rangle)$.

The archaic form of measuring an interval is characterized by the assertions:

1) If $(\neg Ms^3)((s^1 > s^3 \mid \delta)\cdot(s^3 > s^2) \mid \delta))$ then the magnitude of $\langle s^1, s^2 \mid \delta \rangle$ is such that $(\neg Ms^3)((s^1 > s^3 \mid \delta)\cdot(s^3 > s^2 \mid \delta))$.

2) If $(Ms_1)\dots(Ms_n)((s^1 > s_1 \mid \delta)\cdot\dots\cdot(s_n > s^2 \mid \delta))$, where $n \geqslant 1$, the magnitude of $\langle s^1, s^2 \mid \delta \rangle$ is such that $(Ms_1)\dots(Ms_n)((s^1 > s_1 \mid \delta)\cdot\dots\cdot(s_n > s^2 \mid \delta))$.

The interval indicated in 1 precedes the unit of measure, and the fixing of the magnitude of an interval in the way indicated in 2 precedes the measurement of intervals in the assumed system of units. Let α be the name of the interval taken as unit. Then:

A5. If $(Ms_1)\dots(Ms_{n-1})((s^1 > s_1 \mid \delta)\cdot\dots\cdot(s_{n-1} > s^2 \mid \delta))$ and $\langle s^1, s_1 \mid \delta \rangle = \dots = \langle s_{n-1}, s^2 \mid \delta \rangle = \alpha$, then $\langle s^1, s^2 \mid \delta \rangle = n\alpha$ (n units α).

A6. If α and β are units of measure of intervals, $\langle s^1, s^2 \mid \delta \rangle = n\alpha$ and $\langle s^1, s^2 \mid \delta \rangle = m\beta$, then $n\alpha = m\beta$ (the magnitude of an interval does not depend on the selection of the units of measure).

D1. If $(Ms^*)((s^1 > s^* \mid \delta)\cdot(s^* > s^2 \mid \delta))$, then $\langle s^1, s^2 \mid \delta \rangle$ is divisible; if $(\neg Ms^*)((s^1 > s^* \mid \delta)\cdot(s^* > s^2 \mid \delta))$, then $\langle s^1, s^2 \mid \delta \rangle$ is indivisible.

Theoretically every interval greater than zero is infinitely divisible. In practice, however, the process of division stops somewhere: either we find the minimal interval for a given situation or the very concept of interval loses any practical meaning.

T1. The magnitude of an interval depends on the selection of the point of reference in the sense that it is not always true that

$$\langle s^1, s^2 \mid \delta^1 \rangle = \langle s^1, s^2 \mid \delta^2 \rangle;$$

in other words,

$$\sim((\langle s^1, s^2 \mid \delta^1 \rangle = \alpha) \rightarrow (\langle s^1, s^2 \mid \delta^2 \rangle = \alpha)).$$

But the magnitude of an interval does not depend on a change of point of reference, i.e.,

$$(\forall \delta^*) \sim ((\langle s^1, s^2 \mid \delta^* \rangle = \alpha)\cdot(\langle s^1, s^2 \mid \delta^* \rangle \neq \alpha)).$$

If the investigator changes the reference point of the order of s^1 and s^2 this in itself does not mean that the interval between s^1 and s^2 relative to the previous point of reference is no longer the same.

For some intervals it is assumed that they have a finite length. For the other cases the character of the interval and of a series is defined by a system of assertions, like the following:

1) "If the number of elements of a series is finite and all the intervals between its elements have a finite length, then the series has a finite length".

2) "If the number of elements of a series is infinite and there exists the minimal interval, then the series has an infinitely great length".

3) "If at least one of the segments of a series has an infinitely great length, then the series has an infinitely great length; if the series has a finite length, then all its segments have a finite length".

D2. The interval $\langle s^1, s^2 \mid \delta\rangle$ has an infinitely small length if and only if

$$(\forall(\langle s^{*1}, s^{*2} \mid \delta\rangle \downarrow > 0))((\langle s^1, s^2 \mid \delta\rangle \leqslant \leqslant \langle s^{*1}, s^{*2} \mid \delta\rangle)\cdot(\langle s^1, s^2 \mid \delta\rangle > 0))$$

(it is greater than zero and not greater than any interval which is greater than zero).

15. STRUCTURE

D1. We adopt the following definition of structure:

1) if $(s^1 > s^2 \mid \delta)$, then the objects s^1 and s^2 form a binary structure relative to δ;

2) if for any object s^i of a set of objects $s^1,\ldots, s^n$ $(n \geqslant 2)$ one finds another object s^k of this set of objects such that $(s^i > s^k \mid \delta^j)$, then the objects $s^1,\ldots, s^n$ form an n-ary structure relative to some $\delta^1,\ldots, \delta^m$ $(m \geqslant 1)$.

D2. The objects $s^1,\ldots, s^n$, indicated in *D1* are elements of structure.

Structures will be represented by symbols of the type

$$RS(s^1,\ldots, s^n \mid \delta^1,\ldots, \delta^m).$$

It is clear that one object is not a structure, since for any points of reference of an order $\sim(s > s \mid \delta)$. If $(s^1 = s^2 \mid \delta)$, then s^1 and s^2 do not form a structure relative to δ, since $(s^1 = s^2 \mid \delta) \vdash \sim(s^1 > s^2 \mid \delta)\cdot\sim(s^2 > > s^1 \mid \delta)$.

We will not develop here a system of possible concepts describing the types, relations and properties of structures. We limit ourselves to some that are useful for us here.

D3. Two structures $RS(s^1,\ldots, s^n \mid \delta^1,\ldots, \delta^m)$ and $RS(s,\ldots, s_n \mid \delta_1,\ldots,\delta_m)$

are similar if and only if for each pair of elements s^i and s^{i+1} of one of them one can find a pair of elements s_i and s_{i+1} of the other such that

$$((\langle s^i, s^{i+1} \mid \delta^i\rangle \Leftarrow f^1\langle s_i, s_{i+1} \mid \delta_i\rangle)(\langle s_i, s_{i+1} \mid \delta_i\rangle \Leftarrow f^2\langle s^i, s^{i+1} \mid \delta^i\rangle),$$

where f^1 and f^2 are functions, the reverse of each other.

The concepts of length of a series and segment of a series can be extended to structures. We will not introduce a concept for structures, analogous to that of length of a series: in view of the division of labor between logic, physics and mathematics, logic's competence ends here. We will introduce the second concept and it is simple.

D4. The structure $RS(s_1,\ldots, s_m \mid \delta_1,\ldots, \delta_{m-1})$ is a sub-structure of the structure $RC(s^1,\ldots, s^n \mid \delta^1,\ldots, \delta^{n-1})$ if and only if the following are met:

1) the set of objects $s_1,\ldots, s_m$ is a proper subset of objects $s^1,\ldots, s^n$; the set of objects $\delta_1,\ldots, \delta_{m-1}$ is a subset of objects $\delta^1,\ldots, \delta^{n-1}$;

2) every object which is found among $s_1,\ldots, s_m$ is also found among $s^1,\ldots, s^n$.

The second point of *D4* says that not all the objects of the set of objects $s^1,\ldots, s^n$ form a sub-structure of the structure $RC(s^1,\ldots, s^n \mid \delta^1,\ldots, \delta^{n-1})$.

There is a set of methods for fixing that which is called the situation of the object. This is the indication of the series where the object is found as element, of certain series and their directions, of structures, etc. Under all circumstances the situation of an object is always relative to some other objects: if the situation of an object is given then some structure, of which it is an element, is also given. Therefore, the change of situation of an object is always relative.

The elements of series can be structures. Structural series can be continuous and discontinuous, finite and infinite, etc. Not every situation is considered as structure. But every collection of two and more situations is a structure. A description of a structure is made up of a set of sentences which establish the objects, points of reference, series, intervals, etc.

16. RELATION AND FUNCTION

For some relations there are acceptable inferences

$$(\forall s^1)(\forall s^2)(s^1Rs^2)\cdot(s^1 \leftarrow PX) \rightarrow (s^2 \leftarrow PY)$$
$$(\forall s^1)(\forall s^2)(s^1Rs^2)\cdot(s^2 \leftarrow PX) \rightarrow (s^1 \leftarrow PY),$$

where X can vary as something given and the selection of Y depends on the character of X. For example, "s^2 has the magnitude 5α" follows from the sentences "$(\forall s^1)(\forall s^2)$ (s^1 is twice as large as s^2)" and "s^1 has the magnitude 10α" (α is any unit of measure). Such inferences are valid only to the extent that these relations are governed by assertions like

$$(\forall s^1)(\forall s^2)(s^1Rs^2) \equiv (y \Leftarrow f^1(x)) \cdot (x \Leftarrow f^2(y)),$$

where f^1 and f^2 can be identical (e.g., in the case of relations of equality), s^1 occurs unbound in X, s^2 occurs unbound in Y.

CHAPTER EIGHTEEN

PHYSICAL ENTAILMENT

1. EMPIRICAL OBJECTS

Empirical sentences are sentences about empirical objects. If X is an empirical sentence, then $\downarrow X$ is a term designating an empirical object.

2. ORDER OF EVENTS

If objects are events which are fixed in sentences then one can construct ordinal sentences of the type "X, and in relation R to this (i.e., to x) Y". For example, "X, and simultaneously with this (after this, so much time after this, so far from this, etc.) Y". In such sentences R is the designation of the relation of order of events in space and time, of the intervals between them, of the directions of series, of which they are elements, etc.; in short, it designates the position of y in space and time relative to x. We will represent the second part of these sentences by symbols of the type

$$(Rx)\,Y.$$

If X and Y are sentences then $(Rx)\,Y$ is a sentence (*D1*).

The sign Rx is a designation of the place of y in time and space relative to x, and only that. If one uses any other terms of space and time here then they occur in some form in X or Y. For example, in the sentence "In Moscow Z and 24 hours after that in Leningrad U" only the expression "24 hours after that" is Rx, and the corresponding X and Y are "In Moscow Z" and "In Leningrad U".

Rx is often not explicitly formulated. It is usually dropped when one assumes for both events x and y one and the same time-interval and one and the same space, and the order of the events is provided by other parts of the sentences. If Rx occurs in a sentence where it is preceded by X, it is possible that the latter does not occur in Rx: it is replaced by the word "this", "these", etc., referring to the X which precedes Rx. But $(Rx)\,Y$ is an autonomous sentence only if X occurs in Rx.

The truth-value of $(Rx)Y$ is established as follows. One selects a place and time in which X is true and then one establishes the truth-value of Y in a corresponding place relative to x, indicated by R. If Y is then true, $(Rx)Y$ is true (*D2*); if $\sim Y$ is true, then $(Rx)Y$ is false (*D3*). If it is impossible to select a place where X is true, then $(Rx)Y$ is not verifiable (*D4*).

The place of y can be given relative to two and more different events $x^1,\ldots, x^n$. Whence we obtain the sentence

$$((R^1x^1)(R^2x^2))\,Y$$
$$((R^1x^1)(R^2x^2)\ldots(R^nx^n))\,Y,$$

reading "Y in relation R^1 to $x^1,\ldots, R^n$ to x^n". The truth condition of such sentences is the very possibility of the positing of such events relative to $x^1,\ldots, x^n$, given through $R^1,\ldots, R^n$ (the latter can be such that the situation they determine is impossible and then the sentence is not verifiable).

In turn, terms of events are formed from sentences with Rx according to the following rule:

1) if Z is $(Rx)Y$, then z is $\downarrow(y(Rx))$, which reads "y, which is in relation R to x;

2) the terms $\downarrow(y\,((R^1x^1)\cdot\ldots\cdot(R^nx^n)))$ are similarly formed. Now the position of events can be given relative to events of the type examined above and we obtain the sentences

$$(R^1(y(R^2x)))\,Z,\ (R^1(y(R^2x)(R^3z)))\,U,$$

etc. According to the rules of logic of relations and the definitions of $R^1, R^2, R^3,\ldots$ assertions like

$$(R^1(y(R^2x)))\,Z \rightarrow (R^3x)\,Z$$
$$(R^1(y(R^2x)(R^3z)))\,U \rightarrow (R^4x)(R^5z)\,U$$

hold for such sentences.

Sentence X in Rx can have the form $X^1\cdot X^2$, $X^1{:}X^2$, $X^1{:}X^2{:}\ldots{:}X^n$.

An empirical event is always chosen in some spatial-temporal domain. All the other events of this domain are its empirical conditions or the environment. They are implicitly presupposed or partially fixed in special sentences. In the latter case one obtains sentences of the type "X on the condition that V", where V is some set of sentences about empirical objects other than x, about their order, about their order relative to x, etc. Such sentences will be represented as

$$X/v.$$

In the case of X/v there is no logical connection between X and V. However, the establishment of empirical conditions is of great importance in the establishment of the logical connections of sentences about empirical objects.

Sentence X/v is a special case of the previously examined sentences with a complex subject of the type "s is selected on condition that V" (we used the symbol s/v). Therefore, V can be regarded as an isolated portion of the subject, taken out of it. The assertions:

$$X/v \cdot Y/v \equiv (X \cdot Y)/v$$
$$X/v : Y/v \equiv (X : Y)/v$$
$$X^1/v : X^2/v : \ldots : X^n/v \equiv (X^1 : X^2 : \ldots : X^n)/v$$
$$(\alpha \rangle\!| s/v)\,(s/v \leftarrow PX) \equiv ((\alpha \rangle\!| s)\,(s \leftarrow PX))/v$$
$$(X/v \rightarrow Y/v) \equiv (X \rightarrow Y)/v$$

hold for it.

These rules of the exportation of conditions make it possible to simplify sentences by bringing identical signs of conditions into the "context" and by preventing their repetition.

A1. $\sim (Rx)\, Y \dashv\vdash (Rx) \sim Y$

A2. $(Rx)\, Y^1 (Rx)\, Y^2 \dashv\vdash (Rx)\,(Y^1 Y^2)$

A3. $(Rx)\, Y^1 \vee (Rx)\, Y^2 \dashv\vdash (Rx)\,(Y^1 \vee Y^2)$

A4. $(R^1 x)\, Y (s^1 R^1 s^2 \rightarrow s^2 R^2 s^1) \vdash (R^2 y)\, X$

A5. $((R^1 x^1)(R^2 x^2))\, Y \dashv\vdash (R^1 x^1)\, Y (R^2 x^2)\, Y$

A6. $(R(xy))\, Z \dashv\vdash ((Rx)(Ry))\, Z$

A7. $(R(x \vee y))\, Z \dashv\vdash ((Rx) \vee (Ry))\, Z$

A8. $((R^1 x) \vee (R^2 y))\, Z \dashv\vdash (R^1 x)\, Z \vee (R^2 y)\, Z$

3. ORDERED CONJUNCTIONS

Let the symbol $\dot{R}$ be read as "and in relation R" (in particular "and then", "and simultaneously").

D1. $X \alpha \dot{R} Y \equiv ((\downarrow X)\, \alpha R (\downarrow Y))\, XY$

D2. $X^1 \alpha^1 \dot{R}^1 X^2 \alpha^2 \dot{R}^2 \ldots \alpha^{n-1} \dot{R}^{n-1} X^n \equiv (X^1 \alpha^1 \dot{R}^1 X^2) \cdot$
$\cdot (X^2 \alpha^2 \dot{R}^2 X^3) \cdot \ldots \cdot (X^{n-1} \alpha^{n-1} \dot{R}^{n-1} X^n)$

T1. $X\alpha\dot{R}Y \vdash XY$
T2. $\sim(XY \vdash X\alpha\dot{R}Y)$
T3. $\sim(X\alpha\dot{R}Y \vdash Y\alpha\dot{R}X)$

4. PHYSICAL ENTAILMENT

We will call sentences of the form

$$\alpha(X \rightarrow (Rx)\,Y)$$

sentences on physical entailment.

Sentences on physical entailment are obtained as primitive agreements or from other sentences of this kind according to the rules of logic. But one cannot obtain them from the relations of logical entailment according to a schema, analogous to that of quasi-entailment, since they are governed by the assertion:

A1. For any Z if $(Rx)\,Y$ and $X \rightarrow (Rx)\,Y$ do not logically follow from Z, then $(Rx)\,Y$ does not logically follow from $X \cdot Z$.

If sentences about physical entailment are not primitive agreements and are not inferred from other similar sentences, the general schema for constructing them has the following form:

1) $\downarrow X$ is observed;

2) $\downarrow Y$ is observed in the spatial-temporal relation R to $\downarrow X$;

3) that which is indicated in points 1 and 2 happens every time, i.e., for all $\downarrow X$;

4) that which is indicated in points 1 to 3 takes place on conditions V;

5) $(X \rightarrow (Rx)\,Y)/v$ or $X/v \rightarrow (Rx)\,Y/v$ is shorthand for all this.

To what has been said correspond the assertions (the conditional sign has been dropped since it is presupposed throughout):

A2. $(X \rightarrow (Rx)\,Y) \equiv (\forall \downarrow X)((Rx)\,Y)$
A3. $(X \rightarrow (Rx)\,Y) \equiv N \downarrow (Rx)\,Y$

Negation and indeterminacy (which is possible here) are defined by:

A4. $(X \urcorner\!\!\rightarrow (Rx)\,Y) \equiv (\urcorner \forall \downarrow X)((Rx)\,Y)$
A5. $(X \urcorner\!\!\rightarrow (Rx)\,Y) \equiv M \downarrow (Rx) \sim Y$
A6. $(X? \rightarrow (Rx)\,Y) \equiv (?\forall \downarrow X)((Rx)\,Y)$
A7. $(X? \rightarrow (Rx)\,Y) \equiv\, ?M \downarrow (Rx) \sim Y$

Consequences of *A2–A7:*

T1. $(X \to (Rx)\, Y) \equiv \neg M{\downarrow}(Rx) \sim Y$
T2. $(X \neg\!\to (Rx)\, Y) \equiv (\exists{\downarrow}X)((Rx) \sim Y)$
T3. $\sim (X \to (Rx)\, Y) \equiv \sim (\forall{\downarrow}X)((Rx)\, Y)$
T4. $\sim (X\alpha \to (Rx)\, Y) \equiv (X\beta \to (Rx)\, Y){:}(X\gamma \to (Rx)\, Y)$,

where α, β and γ have a meaning analogous to that they had in similar cases above.

T5. $\sim ((X\alpha \to (Rx)\, Y)\cdot(X\beta \to (Rx)\, Y))$,

where α and β are different in any combinations.

A special case of the above schema: for all $\neg x \Rightarrow x$ one observes $\neg x \Rightarrow x$ and one observes $\neg y \Rightarrow y$ in relation R to $\neg x \Rightarrow x$, which is shortened to the sentences $(\neg x \Rightarrow x) \to (R)(\neg y \Rightarrow y)$.

The relation between the sentences considered is defined by

A8. $((\neg x \Rightarrow x) \to (Rx)(\neg y \Rightarrow y)) \to (X \to (Rx)\, Y)$.

But when does the investigator have the right to say “every time”, “for all x” or “for all $\neg x \Rightarrow x$”? One cannot recommend here general rules like those of logical entailment. In asserting this we are not so much mindful of the sad experience of the history of logic as of how things really are: the compelling force of the rules of logical entailment is the compelling force of the agreements of people relative to the properties of logical signs and of the sentential structures containing them. In the case interesting us we have to do with a reflection of the world, which does not depend on convention.

Above all one has to indicate that luck plays an important role in knowledge. In the world there are cases where ${\downarrow}Y$ exists in relation R to ${\downarrow}X$ on any conditions. And if the investigator after some observations accepts $X \to (Rx)\, Y$, the latter becomes an element of scientific knowledge regardless of the lack of any logical foundation for it. There are also cases where ${\downarrow}Y$ always exists in relation R to ${\downarrow}X$ under certain conditions. And these conditions are always given in the experience of the investigator (e.g., the existence of the earth, of the magnetic field, of air, etc.). Moreover, no role is played by whether or not they are known to the investigator. In such cases the fate of $X \to (Rx)\, Y$ is analogous to that mentioned above.

There are some heuristic principles which in practice sometimes produce a positive effect; sometimes they do not. Among them we find the famous inductive methods of Bacon and Mill. To the extent that these heuristic principles sometimes make it possible to obtain true sentences, their use is fully justified. As regards errors, science would be a mediocre affair if scientists did not commit them.

Let us take an example. Let it be the case that in some spatial-temporal domain $\neg x \Rightarrow x$ happens first and then $\neg y \Rightarrow y$ happens. If everything else in the domain in question here remains unchanged, then we can justifiably accept $(\neg x \Rightarrow x) \rightarrow (Rx)\,(\neg y \Rightarrow y)$, where Rx is "after this". However, in the practical use of this principle the constancy of "everything else" is inconceivable and the fate of our sentence depends on how close the constancy of "everything else" is approximated.

5. TRUTH-VALUES

In the case of quasi-entailment the definition of truth-values coincides with a description of the method of constructing sentences. For physical entailment there is no such coincidence. Thus, the following definition is justified: $[X \rightarrow (Rx)\,Y] \leftarrow v^1$ if and only if every time that X is true, Y is true in relation R to x.

But $X \rightarrow (Rx)\,Y$ is accepted as true not, certainly, because one has examined all the cases where X is true and is convinced that Y is true in the corresponding place. If this were the case, the construction of a sentence would be senseless. It would be irreproachable from a logical point of view but it could not be used in new situations where X is true. The sentence is accepted in function of the heuristic considerations we mentioned above. But it can occasionally happen that a thorough enough consideration of one case, where X is true, makes the conditional sentence valid for any number of cases. And even if after being accepted it is confirmed in a great number of cases, this does not mean that the question concerning its truth can be definitely answered even *a posteriori*.

Definitions of the truth-value of physical entailment can be constructed in various ways and in particular as follows (presupposing identity of conditions):

D1. $[X \rightarrow (Rx)\,Y] \leftarrow v^1 \equiv Mx \cdot \neg MxR \sim y$
D2. $[X \rightarrow (Rx)\,Y] \leftarrow v^2 \equiv Mx \cdot ?MxR \sim y$

D3. $[X \rightarrow (Rx)\, Y] \leftarrow v^3 \equiv \sim Mx$
D4. $[X \rightarrow (Rx)\, Y] \leftarrow v^4 \equiv Mx \cdot M(xR \sim y)$
D5. $[X \neg\!\rightarrow (Rx)\, Y] \leftarrow v^1 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^4$
D6. $[X \neg\!\rightarrow (Rx)\, Y] \leftarrow v^2 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^2$
D7. $[X \neg\!\rightarrow (Rx)\, Y] \leftarrow v^3 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^3$
D8. $[X \neg\!\rightarrow (Rx)\, Y] \leftarrow v^4 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^1$
D9. $[X? \rightarrow (Rx)\, Y] \leftarrow v^1 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^2$
D10. $[X? \rightarrow (Rx)\, Y] \leftarrow v^3 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^3$
D11. $[X? \rightarrow (Rx)\, Y] \leftarrow v^4 \equiv [X \rightarrow (Rx)\, Y] \leftarrow v^1 : [X \neg\!\rightarrow (Rx)\, Y] \leftarrow v^1$

In practice it sometimes happens that $M(Rx){\sim}y$ but $X \rightarrow (Rx)\,Y$ is accepted as true since the probability of $(Rx){\sim}y$ is sufficiently small. Therefore, the following definitions are appropriate:

D¹1. $[X \rightarrow (Rx)\, Y] \leftarrow v^1 \equiv Mx \cdot (p(Rx)\, y \geqslant \alpha)$,

where α is a degree of probability sufficiently close to one.

D¹4. $[X \rightarrow (Rx)\, Y] \leftarrow v^4 \equiv Mx \cdot (p(Rx)\, y \leqslant \beta)$,

where β is a degree of probability sufficiently close to zero.

6. DEDUCTIVE PROPERTIES OF PHYSICAL ENTAILMENT

Axiomatic schemata *AI*:

1. $(X \rightarrow (Rx)\, Y) \dashv\vdash (\forall \downarrow X)((Rx)\, Y)$
2. $(X \rightarrow (Rx)\, Y) \dashv\vdash N \downarrow ((Rx)\, Y)$
3. $\neg (X \rightarrow (Rx)\, Y) \dashv\vdash (\neg \forall \downarrow X)((Rx)\, Y)$
4. $\neg (X \rightarrow (Rx)\, Y) \dashv\vdash M \downarrow ((Rx) \sim Y)$
5. $?(X \rightarrow (Rx)\, Y) \dashv\vdash (? \forall \downarrow X)((Rx)\, Y)$
6. $?(X \rightarrow (Rx)\, Y) \dashv\vdash ?M \downarrow ((Rx) \sim Y)$

Axiomatic schemata *AII*:

1. $(X \rightarrow (Rx)\, Y)\, X \vdash (Rx)\, Y$
2. $(X \rightarrow (R^1x)\, Y)(s^1R^1s^2 \rightarrow s^2R^2s^1) \vdash (\sim Y \rightarrow (R^2 \sim y) \sim X)$
3. $(X \rightarrow (R^1x)\, Y)(Y \rightarrow (R^2y)\, Z)((s^1R^1s^2)(s^2R^2s^3) \rightarrow (s^1R^3s^3)) \vdash \vdash (X \rightarrow (R^3x)\, Z)$
4. $(XY \rightarrow (R^1x)(R^2y)\, Z) \dashv\vdash (X \rightarrow ((R^1x)\, Y \rightarrow (R^2y)\, Z))$
5. $(X \rightarrow (Rx)(YZ)) \dashv\vdash (X \rightarrow (Rx)\, Y)(X \rightarrow (Rx)\, Z)$

6. $(X \to (R^1x)\, Y)\,(X \to (R^2x)\, Z)\,((s^1R^1s^2)\,(s^1R^2s^3)) \to$
   $\to (s^1R^3(s^1, s^2)) \vdash (X \to (R^3x)\,(YZ))$
7. $((X \vee Y) \to (R^1x)\,(R^2y)\, Z) \dashv\vdash (X \to (R^1x)\, Z)\,(Y \to (R^2y)\, Z)$
8. $(X \to (Rx)\,(Y \vee Z)) \dashv\vdash (X \to (Rx)\, Y) \vee (X \to (Rx)\, Z)$
9. $((X \to (R^1x)\, Y) \vee (X \to (R^2x)\, Z))\,((s^1R^1s^2)\,(s^1R^2s^3)) \to$
   $\to (s^1R^3(s^2, s^3)) \vdash (X \to (R^3x)\,(Y \vee Z))$
10. $(X \to (Rx) \sim Y) \vdash \sim (X \to (Rx)\, Y)$
11. $(X \to (Rx)\, Y)\,(Y \to Z) \vdash (X \to (Rx)\, Z)$

7. PHYSICAL ENTAILMENT AND FUNCTIONS

Let us construct the sentences

$$X^1 \to (Rx^1)\, Y^1,\ X^2 \to (Rx^2)\, Y^2,\ X^3 \to (Rx^3)\, Y^3, \ldots .$$

Let $P^1, P^2, P^3, \ldots$ be predicates of $X^1, X^2, X^3, \ldots$ respectively, and Q^1, Q^2, $Q^3, \ldots$ predicates of Y^1, Y^2, $Y^3, \ldots$ Further, suppose that it was found that

$$Q \Leftarrow f(P),$$

where $P^i \in KP$ and $Q^i \in KQ$. Then instead of the initial set of sentences we can adopt

$$X \to (Rx)\, Y,$$

where $\downarrow x^i \in K \downarrow x$, $\downarrow y^i \in Ky$, $\downarrow y \Leftarrow f(\downarrow x)$. Shorthand for this will be

$$\downarrow (Rx)\, Y \Leftarrow f(x).$$

Here f is selected so that for each X^i there is a Y^i constructed so that

$$X^i \to (Rx^i)\, Y^i.$$

If every X^i is $X^i_1 \cdot \ldots \cdot X^i_n\, (n \geqslant 1)$, then the function takes the form

$$\downarrow R(x_1 \cdot \ldots \cdot x_n)\, Y \Leftarrow f(x_1, \ldots, x_n).$$

Because of the method of their construction the sentences $(Rx)y \Leftarrow f(x)$ have the property that

$$(\downarrow (Rx)\, Y \Leftarrow f(x)) \cdot X^i \to (Rx^i)\, Y^i.$$

Moreover, the sentence $(Rx^i)\, Y^i$ can be verified (which means obtained) independently of X^i and $(\downarrow Rx)\, Y \Leftarrow f(\downarrow X)$.

The sentence $X \rightarrow (Rx)Y$ can be regarded as a special case of the sentence $\downarrow(Rx)Y \Leftarrow f(\downarrow X)$, where the classes of events $\downarrow X$ and $\downarrow Y$ are characterized only by the fact that $\downarrow X \in K \downarrow X$ and $Y \in K \downarrow Y$, i.e., they are not differentiated in type.

8. TWO-VALUED AND MANY-VALUED FUNCTIONS

The functions of two-valued and many-valued sentential logic can be interpreted as special cases of the functions examined above (let us call them empirical functions). These functions have the following properties: given is a class of objects and a class of attributes which can be ascribed to them; each object of such a kind has one of these attributes but only one at a time (the attributes exclude each other). The sentential variables of logic are interpreted as such objects, their values as possible attributes of these objects, and the sentential functions as the empirical dependence of the states of some of these objects on the states of other objects. The sign Rx disappears since we assume simultaneity or sequentiality of the events.

As an example we will take the functions of two-valued logic. Let there be a class of objects such that they can have the P^1 and P^2 and only these. For any objects of the given class the following assertions hold:

1) $\sim(s \leftarrow P^1) \equiv (s \neg \leftarrow P^1)$, $\sim(s \leftarrow P^2) \equiv (s \neg \leftarrow P^2)$
2) $(s \leftarrow P^1) \rightarrow \sim(s \leftarrow P^2)$, $\sim(s \leftarrow P^1) \rightarrow (s \leftarrow P^2)$.

One can accept

$$(s \leftarrow P^1) : (s \leftarrow P^2)$$

instead of the second assertion.

Different sorts of dependence can exist between objects of this class; for example:

D1. Objects s^1 and s^2 "negate" each other if and only if

$$(s^1 \leftarrow P^1) \rightarrow (s^2 \leftarrow P^2), (s^2 \leftarrow P^1) \rightarrow (s^1 \leftarrow P^2),$$
$$(s^1 \leftarrow P^2) \rightarrow (s^2 \leftarrow P^1), (s^2 \leftarrow P^2) \rightarrow (s^1 \leftarrow P^1).$$

T1. If s^1 and s^2 "negate" each other then

$$(s^1 \leftarrow P^1) \rightarrow \sim(s^2 \leftarrow P^1), (s^1 \leftarrow P^2) \rightarrow \sim(s^2 \leftarrow P^2),$$

$$(s^2 \leftarrow P^1) \rightarrow \sim (s^1 \leftarrow P^1), (s^2 \leftarrow P^2) \rightarrow \sim (s^1 \leftarrow P^2).$$

T2. $((s^1 \leftarrow P^1):(s^2 \leftarrow P^1))\cdot((s^1 \leftarrow P^2):(s^2 \leftarrow P^2))$

D2. Objects s^1 and s^2 form a "conjunction" relative to s^3 if and only if

$$(s^1 \leftarrow P^1)\cdot(s^2 \leftarrow P^1) \leftrightarrow (s^3 \leftarrow P^1)$$
$$(s^1 \leftarrow P^1)\cdot(s^2 \leftarrow P^2) \rightarrow (s^3 \leftarrow P^2)$$
$$(s^1 \leftarrow P^2)\cdot(s^2 \leftarrow P^1) \rightarrow (s^3 \leftarrow P^2)$$
$$(s^1 \leftarrow P^2)\cdot(s^2 \leftarrow P^2) \rightarrow (s^3 \leftarrow P^2)$$
$$(s^3 \leftarrow P^2) \rightarrow ((s^1 \leftarrow P^1)\cdot(s^2 \leftarrow P^2):(s^1 \leftarrow P^2)\cdot(s^2 \leftarrow P^1):$$
$$:(s^1 \leftarrow P^2)\cdot(s^2 \leftarrow P^2))$$

The other functions are interpreted in a similar way. This is a truth-value matrix in different form, and nothing else.

9. EMPIRICAL CONNECTIONS

D1. We will say that $\downarrow X$ and $\downarrow Y$ which figure in $X \rightarrow (Rx)Y$ and $\downarrow (Rx)Y \Leftarrow f(\downarrow X)$, are in empirical connection (form an empirical connection). We will also say that $\downarrow X$ is in empirical connection with $\downarrow Y$ (and $\downarrow Y$ with $\downarrow X$). Expressions which contain instead of $\downarrow X$ and $\downarrow Y$ the s^1 and s^2 which occur in them have a similar meaning. The character of the connection of the events $\downarrow X$ and $\downarrow Y$ is given by the signs R and f and the character of the connection of the objects s^1 and s^2 is defined in addition to this by the remaining parts of X and Y. Events $\downarrow X$ and $\downarrow Y$ (objects s^1 and s^2) are elements of a connection.

Thus the answer to the question "how are the events $\downarrow X$ and $\downarrow Y$ (or objects s^1 and s^2) connected?" is formed by the sentences $X \rightarrow (Rx)Y$ and $\downarrow (Rx)Y \Leftarrow f(\downarrow X)$. However, in everyday science such questions are often answered by naming other events and objects which have the following properties: one can construct sentences $Z^1, \ldots, Z^n$ $(n \geqslant 1)$ such that

$$(X \rightarrow (R^1x)\, Z^1)\cdot(Z^1 \rightarrow (R^2z^1)\, Z^2)\cdot \ldots \cdot(Z^{n-1} \rightarrow (R^n z^{n-1})\, Z^n)\cdot$$
$$\cdot(Z^n \rightarrow (R^{n+1}z^n)\, Y) \rightarrow (X \rightarrow (Rx)\, Y).$$

In this case one can use the expression "mechanism" of connection (*D2*). Thus, the answer to the question as to what are the mechanisms of the connection of $\downarrow X$ and $\downarrow Y$ (correspondingly of s^1 and s^2) is only the antecedent of the above proposition. Sometimes one gives a simple answer by listing the objects found in Z^i. But this changes nothing.

D3. A set of events $\downarrow X^1, \ldots, \downarrow X^n$ (objects $s^1, \ldots, s^n$; $n \geqslant 2$) forms an empirical system of connections if and only if for every $\downarrow X^i$ (s^i) of this set one finds at least one $\downarrow X^k$ (s^k) of the same set of events (objects) such that it is empirically connected with it. The connections which form the system in question are its elements.

Those empirical systems of connections which are really studied by science are characterized by a more narrow notion of an isolated empirical system of connections (for simplicity we will speak of an isolated system). One here has in mind the following:

D4. A set of events (objects) is given in some (no matter how it is limited) spatial-temporal domain; and one considers a system made up only of these events (objects), i.e., all other events (objects) are left aside; if one does consider other events (objects) then the system considered is "expanded" (relative to the initial system) but still isolated. And this is natural since "one cannot encompass the unencompassible". Single connections are now regarded as elements of the isolated systems. For all elements of a system the same conditions are presupposed.

D5. A connection is immediate if for it (in the system in question, of course) one cannot indicate other connections of the same system which form its mechanism; otherwise, it is indirect.

D6. A connection is simple if it cannot be presented as a system of two or more different connections; otherwise it is complex.

The basic problem of the study of empirical systems is the discovery of simple and immediate connections such that from sentences about them we can obtain sentences about any complex and indirect connections of the same system, and to formulate the rules for doing this. Two operations are of interest here: the isolation of single connections (analysis of the system) and combining (synthesis) them into complex connections.

Analysis of a system is characterized by the notion of isolated connection (connection in its pure form; connection as such, etc.), which we define as follows (*D7*).

For simplicity we assume that R, R^1, R^2,... somehow fix the position of events which figure in the consequent relative to events which figure in the antecedent. Let

$$\downarrow (R)\, Y \Leftarrow f(x_1, \ldots, x_n)$$

$$\downarrow(R)\,(Y_1\cdot\ldots\cdot Y_m)\Leftarrow f(x_1)$$
$$\downarrow(R)\,(Y_1\cdot\ldots\cdot Y_m)\Leftarrow f(x_1,\ldots,x_n)$$

be sentences, formulating connections of the system in question. We assume that in every domain of investigation one can establish $X_1^*,\ldots,X_n^*$ such that $x_1^*\in Kx_1,\ldots,x_n^*\in Kx_n$. The sentences

$$\downarrow(R)\;Y\Leftarrow f^1(x_1,x_2^*,x_n^*)$$
$$\downarrow(R)\;Y\Leftarrow f^2(x_2,x_1^*,\ldots,x_n^*)$$
$$\cdots\cdots\cdots\cdots$$
$$\downarrow(R)\;Y\Leftarrow f^n(x_n,x_1^*,\ldots,x_{n-1}^*)$$

are called sentences about isolated connections (about connections in the pure form) relative to

$$\downarrow(R)\;Y\Leftarrow f(x_1,\ldots,x_n).$$

Similarly for $Y_1,\ldots,Y_m$; similarly for $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$ together. The sentences X_{j*} and Y_{j*} are conditions of isolation. Such are, for example, sentences saying that the corresponding objects do not change, that they do not exist, that they do not influence other objects, etc.

By synthesis of isolated connections into complex ones we mean (*D8*) the search for rules making possible the sentences

$$(\downarrow(R)\;Y\Leftarrow f^1(x_1,x_2^*,\ldots,x_n^*))\cdot(\downarrow(R)\;Y\Leftarrow$$
$$\Leftarrow f^2(x_2,x_1^*,\ldots,x_n^*))\rightarrow(\downarrow(R)\;Y\Leftarrow$$
$$\Leftarrow f^3(x_1,x_2,x_3^*,\ldots,x_n^*))\,;$$

similarly for three and more arguments; similarly for two and more $Y_1,\ldots,Y_m$; similarly for $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$ together. Discovery of such rules involves acceptance (explicit or implicit) of extra-logical assumptions relative to the system.

Connections are not structures. They are merely means (possibility, necessity, etc.) of forming the structures of some area of the world. This is indicated by the fact that they exist in a different sense. For the existence of a relation the existence of the first object in the order (the realization of the first event in the order) is necessary.

10. CONTRAFACTUAL SENTENCES

Contrafactuals are sentences of the type "If X had been (had not been) then Y would have been (have not been)". They are shorthand for

$$X \cdot (Rx)\, Y \cdot (\sim X \urcorner\!\rightarrow (R \sim x) \sim Y).$$

11. SENTENCES ON CONNECTIONS AND INDIVIDUAL EVENTS

Sentences hold for individual events either in function of the general principle

$$(X \rightarrow (Rx)\, Y)\,(t^2/t^1) \cdot (t^2 \in Kt^1) \vdash (X \rightarrow (Rx)\, Y)$$

or in function of the reasoning by means of which we get Y from X and some set of sentences Z which are considered true.

CHAPTER NINETEEN

THEORIES

1. THEORY

In science terms and sentences are united in complexes. The logical theory of scientific knowledge usually considers such of them as are formed according to the rules of logical entailment and of substitution of terms; they are called theories. Here we will take up in brief form only a few of the fundamental questions concerning theories.

The term "theory" is used in different senses. We will take up only one of them here. Let there be a domain of science *S*. Let *A* be a non-empty set of sentences relative to *S*, and *B* some (possibly empty) set of sentences also relative to *S*. Finally, let *Th* be a non-empty set of universal sentences.

D1. If from *Th* and true sentences relative to *B* one gets with sufficient regularity true sentences relative to *A*, and the rules of logic are sufficient for obtaining them, then we will say that *Th* plays the role of theory (is a theory) relative to *A* and *B*.

Obtaining the sentences *A* from *B* without *Th* in a purely logical way is impossible or is possible only in a small number of cases (or in cases without value). Otherwise *Th* loses practical meaning.

Since *Th* makes it possible to obtain sentences *A* in a purely logical way it can be considered part of the theory of inference. But this is an extra-logical phenomenon (we exclude here theories occurring within logic itself).

We have defined theory in such a way that even isolated universal sentences taken singly can play the role of theory provided they are regularly used to obtain sentences. In the concrete sciences not every collection of universal sentences deserves the name of theory. So there are some extra-logical requirements for theory.

We omit questions having to do with the formation and development of theory (this is a creative process which depends on concrete conditions in science) and with the functions of theories in knowledge (explanation, confirmation, prediction, systematization, abbreviation, explication, etc.). We will only make some remarks on the structure of theories.

D2. The sentences included in a given theory are initial (primitive) or derivative. The primitive are simply taken as given; the derivative are obtained from the primitive.

Among the primitive assertions of a theory one finds:

1) assertions which can be (and are) obtained and verified independently of the construction of the theory in question and of its other assertions (in particular this can be the results of observations);

2) assertions which are transformations of the definitions of primitive terms;

3) assumptions.

D3. The terms which figure in a given theory are also divided into primitive and derivative. The primitive terms are not definable one by means of another and figure in primitive assertions; the derivative terms are those defined through the primitive ones.

Included among the primitive terms are:

1) terms whose meaning can be established independently of the theory (i.e., of the other terms in it); these can be considered primitively clear, explained by examples, or defined by means of terms from other domains of science.

2) terms whose sense is defined by primitive assertions.

In distinguishing primitive and derivative terms and sentences we are not asserting that all of them are given at once in the construction of a theory. A theory can be so established that there are no *a priori* limits to the number of primitive terms and sentences and the latter can be introduced as needed in the course of the development of the theory. These theories can be called open as distinguished from the closed where the primitive base is limited (*D4*).

D5. If $Y^1, \ldots, Y^r$ are primitive assertions of a given theory and $Y^1 \cdot \ldots \cdot Y^r \vdash Z$, then Z is an intrinsic consequence of the primitive assertions of the theory. If $\sim (Y^1 \cdot \ldots \cdot Y^r \vdash Z)$, $\sim (W \vdash Z)$, $\sim (Y^1 \cdot \ldots \cdot Y^n \vdash W)$ and $Y^1 \cdot \ldots \cdot Y^r \cdot W \vdash Z$, then Z is an extrinsic consequence of the theory.

D6. If $t_1, \ldots, t_k$ are primitive terms and t^i is defined by them in such a way that no other terms are used, then t^i is an intrinsically derived term of the theory. If in addition to $t_1, \ldots, t_k$ one needs for the definition of t^i some set of terms which are not definable by $t_1, \ldots, t_k$, then t^i is an extrinsically derived term of the theory.

Whether or not extrinsic universal consequences and extrinsically derived terms are included in the structure of a given theory is not of great importance. In practice science uses both the extrinsic and intrinsic universal consequences of primitive assertions (and also extrinsically derived terms).

Axiomatic theories are closed theories. A special case of axiomatic theory is one whose primitive assertions are definitions of primitive terms. A special case of theory is a theory which aims at the explication of the concepts of the given domain of science. The so-called hypothetical-deductive theories are just such special cases of theory in our sense of the term.

The rules for the introduction of derivative terms and for the obtaining of derivative assertions (the development of theory) are above all rules of logic. But two remarks are necessary. First, in logic for the definition of a class of rules of inference one cannot presuppose that these very rules are given. This is not necessary: one only need show that there are some rules which can be used to get new rules from the rules of logic already given. This is why one fixes with precision in the construction of a series of logical theories the rules to derive assertions from the primitive ones. Second, in a series of sciences (mathematics, physics, linguistics, etc.) some theories are constructed by combining their primitive assertions with some logical calculus. This step is justified only to the extent that one has to define the class of rules of inference precisely since the intuitive versions of these are not always reliable. The logical calculus does not here become a part of the theory in question (for example, the calculus of predicates is not a part of arithmetic); it remains merely a method for developing it.

Among the rules of the development of theory we also find the rules for operating with terms which have a general character (general for all sciences), especially the rules of mathematics. Since they are universal they can be considered on the same plane as the rules of logic, i.e., as subjective means of obtaining knowledge, as habits of a special type. But, their definitions are rightly seen as assertions which serve for obtaining the extrinsic consequences of a theory.

D7. If $Y^i,\ldots, Y^k$ are the assertions of a given theory, and Z^1 is an assertion obtained in the given domain of investigation and Z^2 is obtained from $Y^i,\ldots, Y^k, Z^1$ (according to the rules of logic and mathematics), then the obtaining of Z^2 is its being obtained from Z^1 by means of the given theory (we assume that it is impossible to obtain Z^2 in the same way from Z^1 only).

2. THEORETICAL ASSUMPTIONS

Among the primitive assertions of a theory one can find assumptions which either cannot be proved in themselves or contradict empirical fact. Their acceptance is justified by the fact that they make possible deduction and the obtaining of the necessary conclusions in the science in question. These assumptions are essentially abstract, i.e., they represent the decision not to take into account any attributes of the objects studied or to take into account only certain attributes. For example, all the objects of a given class can be taken as differing only in position in space, as absolutely independent of each other, etc. Obviously, the intentions of the investigator do not have a truth-value. They cannot be confirmed or refuted. They can only be justified or rejected in function of their consequences. But inferences cannot be made from intentions. Therefore, they are given sentential form and considered true. And even though they themselves might be false, indeterminate or even unverifiable, their consequences can be considered true.

Among the theoretical assumptions we find the universal assertions, usually called "general laws of nature". For example, "Every qualitative change is the result of quantitative changes", "Nature is continuous", "All natural processes do not end in an instant (there is some inertia)" (which is the basis for extrapolation), "Nature does not make leaps" (which is the basis for interpolation), "Everything happens in the form of leaps", "Sooner or later all natural processes die out (fade)", "Sooner or later all progress reaches a limit", "All objects in nature are ordered", "Chaos reigns in nature", etc. Everything we said above applies to these assertions if they are formulated not as partial assumptions of relative import but as universal.

Theoretical assumptions widen the scope of deduction. The sentences obtained in this way are true over a certain range.

3. PROPERTIES AND RELATIONS BETWEEN THEORIES

The properties and relations between theories are the properties and relations between the sentences and terms which form these theories. Therefore, the general considerations of logic on this subject are quite trivial and give the impression of being prefatory to something which has not yet been invented.

D1. A theory is considered consistent if and only if $\sim(Y^1 \cdot \ldots \cdot Y^r. W \rightarrow Z \cdot \sim Z)$, where $Y^1, \ldots, Y^r$ are its primitive assertions and W is true or lacking. Logically inconsistent theories exist and are used in science. This is possible because they contain consistent fragments which make it possible to obtain true sentences. But the detection of logical inconsistency in a theory is usually a stimulus for its perfection, i.e., for the construction of a consistent theory.

D2. A primitive assertion Y^i of a given theory does not depend on the other primitive assertions $Y^{k1}, \ldots, Y^{kl}$ if and only if $\sim(Y^{k1} \cdot \ldots \cdot Y^{kl}. W \rightarrow Y^i)$, where $\sim(W \rightarrow Y^i)$ and W is true or empty. A primitive term does not depend on the other primitive terms of a theory if and only if it is not defined by means of them. Detection of the dependence of some primitive terms (assertions) on others is a stimulus to "minimization" of the primitive elements of the theory. However, dependence of this type does not lead to phenomena like logical inconsistency.

As we see, this conception of the consistency and independence of the elements of a theory is a "weakening" of the concepts adopted by logic, in the interests of a description of the constructions called theories made up of sentences and terms, found in science. The same is true of completeness. Here the following cases are possible (*D3*):

1) There is a set of sentences $X^1 \ldots X^m$ and the theory is considered complete or incomplete (with some additional descriptions like "intuitively", "empirically", "*a posteriori*", etc.) depending on whether or not all $X^1, \ldots, X^m$ can be obtained by means of this theory (degrees are possible depending on if one has in mind only the internal or simply any of the consequences of the theory);

2) there are some *a priori* requirements which have to be satisfied by the sentences of a given domain of science; a given theory is considered complete or incomplete on the basis of whether or not all the sentences

meeting these requirements are obtained by means of this theory (with some limitations like "deductively", "*a priori*", etc.).

Theories are related in different ways. Their relations are partly defined as relations of classes obtained in them and by means of their sentences; they are generalizations of the relations between axiomatic systems, usually studied by logic.

D4. $Th^1 \subset Th^2$ if and only if for any Z "If $Th^1 \rightarrow Z$, then $Th^2 \rightarrow Z$".

T1. If $Y^1, \ldots, Y^n$ are the primitive assertions of Th^1 and $X^1, \ldots, X^m$ are the primitive assertions of Th^2 and $X^1 \cdot \ldots \cdot X^m \rightarrow Y^1 \cdot \ldots \cdot Y^n$, then $Th^1 \subset Th^2$.

D5. $Th^1 \subset \supset Th^2$ (are equivalent) if and only if $(Th^1 \subset Th^2) \cdot (Th^2 \subset Th^1)$.

The other relations (of union, of compatibility, etc.) can be similarly defined. But this does not exhaust the question of inter-relations between theories. In particular, the following types of relations are of interest:

D6. Let there be $Th^1 \rightarrow X^1$ and $Th^2 \rightarrow X^2$. Both X^1 and X^2 occur in the range of truth relative to x. But one of them is rated as more exact, less exact or just exact. Similar comparisons are possible for the other consequences of Th^1 and Th^2 and their aggregate constitutes a summary evaluation of the comparative precision of the theories. More complex relations result from the comparison of the sets of consequences and of their degrees of exactness.

D7. Let t^1 be a primitive term of Th^1 and t^2 of Th^2. Let it be the case that $Kt^1 \subset \cdot Kt^2$ (or $t^2 \rightharpoonup t^1$). Then Th^1 will be considered a special case of Th^2 if they are not otherwise distinguished.

Theory Th^1 can be obtained by adding to the primitive assertions of Th^2 an additional assertion, independent of them. It is clear that $Th^2 \subset Th^1$. This can be done implicitly by the introduction of an additional term. Conversely, one theory can be obtained from another by eliminating some elements of its primitive base.

D8. Let $s^1, \ldots, s^k$ be terms of Th. The value-range of Th is $Ks^1 \cup \ldots \cup Ks^k$. The relations between the sets of consequences of Th^1 and Th^2 are explained in accord with the general principles of logic.

Theories are also compared according to sets and according to the character of sentences (degree of exactitude, of approximation, etc.) which they make it possible to obtain in a given science. There are cases where one can find among the primitive assertions of Th^1 and Th^2 such X and Y, respectively, that $\sim (X \cdot Y)$ and nevertheless these theories provide similar sets of sentences in a given science. This is due to the fact that it is possible to have empirical sentences $X \rightarrow (R)Z$ and $Y \rightarrow (R)Z$ such that $\sim (X \cdot Y)$.

4. THEORY AND EXPERIENCE

Theory is formulated in order to gain knowledge without having recourse to empirical investigation (it replaces it). This is particularly important in the case of prognoses when one cannot conduct empirical investigations. But the very formation of theory requires some results of empirical investigation. What is more, in the process of the development of theory one makes use of the most diverse modes of investigation right up to observation and experiments. In the final analysis the coincidence of sentences obtained by means of the theory with empirical data confirms the theory or dooms it to rejection as ineffective or even dangerous (leading to error).

If one runs into cases where sentences of the theory or sentences obtained with its help do not coincide with the results of empirical investigations (they fall outside the range of truth), the situation thus created does not form a logical contradiction.

After it is constructed, a theory is used as a partial theory of inference from the sentences of the given domain of science. One uses the expressions "theoretical confirmation", "theoretical establishment", "experimental confirmation of theory", etc. All expressions like these fix (in a more or less explicit form) the inter-relations between sentences which are obtained theoretically (in and through theory) and empirically.

5. THEORY AND FORMAL SYSTEM

A formal system is constructed as follows: a set of primitive objects is given; one designates other objects which make it possible to obtain new objects, and the rules of forming the latter; one defines a subset of objects which are somehow distinguished, and one refers to rules for obtaining other designated objects from them. This is not theory since the formal system does not have terms and sentences (in our sense of the terms). Theory can be obtained only by an interpretation of a formal system with its objects being considered terms, sentences and logical signs.

Moreover, the interpretation is selected in such a way that one obtains a theory which satisfies the requirements of truth.

By "formalization of a theory" one often means completely different things: 1) abstracting from the meaning of the terms of the theory in order to examine its logical consistency; 2) axiomatization; 3) construction of a formal system such that its interpretation provides a theory equivalent to that which is given.

The relation of model to original can be established between theories, between theories and formal systems, and between formal systems themselves.

6. NON-DEDUCTIVE PRINCIPLES

The so-called non-deductive principles, which are usually not explicitly formulated, play an essential role in the construction of theory. These are the principles which govern the sequence of consideration of objects in the domain in question, the selection of the starting point for this process, the discovery of primitive concepts and assertions, etc. These matters are weakly developed in logic. This is due not so much to neglect on the part of logicians, as to the fact that the solution of such problems depends on the peculiarities of the object-range under study (and of the sciences involved) which are the responsibility of the sciences in question and not of logic. The general schemata and recommendations which logic can at present offer in this regard have a very low heuristic value.

CHAPTER TWENTY

LOGIC AND ONTOLOGY

1. ONTOLOGICAL ASSERTIONS IN LOGIC

In a series of cases the laws of logic can take the form of assertions not about the properties of terms and sentences but about objects to which the terms and sentences relate (i.e., the form of ontological assertions). For example, the assertions "From the sentence $\sim(XY)$ follows the sentence $\sim X \vee \sim Y$" can take the form "If the situation $\sim(XY)$ happens, then the situation $\sim X \vee \sim Y$ happens".

Such an ontologization of the assertions of logic is connected, however, not with the nature of these laws but with the habit of relating the content of sentences to the corresponding objects and with convenience of language. The mere posing of the question concerning the basis of such assertions makes clear that they result from the definitions of the logical operators they contain.

Further, using certain rules, ontological assertions can be obtained from logical ones. Thus, from the meta-assertion $X \vdash Y$, made up of the subjects $[X]$ and $[Y]$ and the two-place predicate $\vdash$, one obtains the conditional sentence $X \rightarrow Y$, made up of the sentences X and Y and the logical operator $\rightarrow$.

The particularity of such ontological conclusions from logical laws is that they are *a priori*, they do not depend on experience, they are true on a purely logical basis. Thus, if Y logically follows from X then $X \rightarrow Y$ is true irrespective of the concrete content of X and Y.

Further, one accepts in logic assertions which have a directly ontological form. Such are the assertions $\sim(X \sim X)$, $X \vee \sim X$, for example. However, in this case such assertions are accepted not because the world around us is structured that way (i.e., as generalizations of the results of observation) but strictly because they are consequences of definitions of their logical signs or themselves are parts of an implicit definition of these signs. Thus, in the definitions of the signs $\sim$, $\cdot$, $\vee$, etc., the truth-tables show these assertions to be always true. If, however, they are taken as implicit definitions of these signs, then they literally mean the following: the signs $\sim$, $\cdot$, $\vee$, etc., by our decision and by stipulation are such that the assertions $\sim (X \sim X)$, $X \vee \sim X$, etc., will be true for any X. But in this case one assumes the connection of a given section of logic with others; in particular it is assumed that the class of such assertions is identical with the class of tautologies with analogous signs.

And when we assert, for example, that there has at no time and in no place in the world been a situation $\sim X \cdot X$, then our conviction is based not at all on the fact that we studied the world at all times and in all places, but on the fact that we construed the signs $\sim$ and $\cdot$ in this way. It is simply that in our language acceptance of the possibility of $\sim X \cdot X$ is inconsistent with the signs $\sim$ and $\cdot$, and nothing more.

Of course, the practice of knowledge obliges people to learn the use of a certain type of logical sign. In practice one meets situations where one object excludes another, some objects coexist with others, etc.; these serve as point of departure for the introduction of the corresponding logical signs. But this does not affect the fact that these very objects are products of peoples' creativity, that they have the properties indicated above.

2. PARADOXES OF MOTION

In this connection it is of interest to return to the paradox of motion.

Most of the time one answers the question "Can a physical body be and not be in the same place at the same time?" negatively. And in most cases the motives for this negative answer deserve criticism.

A physical body cannot be and not be in the same place at the same time because the world is like that – this is the usual answer. In fact, in our experience we do not find cases contradicting this answer. And they are never met. But the reason for this is basically different from the reason why we do not meet horses with ten horns. The reason is to be found in the fact that we use the signs "and" and "not", and that the sentence "a physical body is at a given time in a given place" is a sentence. No special wisdom is required. The complexity of the problem is a function of its triviality (rather, of the refusal to recognize its triviality).

The basis for the negative answer to this question can sometimes be found through analysis of the meaning of the expression "to be in a place". The idea that the source of ontological assertions in logic could be the definition of the terms is completely justified. And we will now examine such a case. But in this instance the meaning of the expression "to be in a place" does not play any role.

We will take another question, closely connected with the paradoxes of motion: can a physical body be simultaneously in two different places? This is usually answered negatively and this is motivated by reference to the laws of the world or to the laws of logic. Here the essential point is the meaning of the expression "different places".

Two regions of space are considered different ("different places") if they do not have common "points" – they do not intersect. But what is it about the "points" that "they do not intersect?" The only escape from the difficulty is reference to physical bodies. The definition has to take the form: two regions of space are different (are "different places", do not intersect, etc.), if and only if for any physical body there is the assertion that if the body is in one region of space then it is not at the same time in another one.

Thus, the negative answer to the question is the result of the definition of the expression "different places". It would be enough to adopt another definition in order to get a positive answer. Thus, if in this definition we replace the universal quantifier (the expression "for any physical body") with an existential quantifier or a sign of possibility (the expression "a physical body exists (is possible), for which") then different places can have common "points" and it is possible to have cases where a body is simultaneously in different places (in this sense). We introduce this reference to another definition only to show that in this case the heart of the matter is the analysis of the sense of the term "different places".

In a word, logic asserts nothing *a posteriori* about the world. But assertions about the world are themselves certain structures of signs (terms, sentences, logical signs). And it is possible to construe assertions as true because of the properties of the signs and not because of the properties of the world. These assertions form the logical limits of knowledge in any science, conditioned by the properties of the semiotic apparatus of reflection (just as the possibilities of every tool are limited by the properties of the tool itself).

3. SPACE AND TIME

There is a more precise and profound relation of logic and ontology. This is connected with the fact that many physical terms ("source", "end", "space", "time", "cause", etc.) can be made more precise through the terms of logic with all the resulting consequences. We will briefly examine the terms "space" and "time".

It is impossible to define the terms "space" and "time" with expressions like "Space (time) is something or other". Other definitions are needed: namely, ones which establish the values of expressions containing the terms "right", "left", "farther", "closer", "distance between", "simultaneously", "before", "later", "after so much time", etc.; i.e., terms which designate the spatial and temporal orders, intervals, directions of orders, etc. And this can be done by a suitable interpretation of the corresponding expressions of the logic of relations, which we examined above.

A representation of space is above all (and ultimately) a representation of the ordering of empirical objects. And one has to assume in the representing subject the presence of a capability for the activities which will permit establishing the spatial order of objects (turning of the head, body, eyes; moving toward and away from the object, etc.). These activities of the reflecting subject are carried out in the reflection of the objects. The latter are selected in a sufficiently restricted time-interval (they coexist). So there is no problem in establishing the temporal order of some objects for the reflecting subject (it is determined by his activities). The spatial order of other objects is defined through that of those which are already given. Measurement of the spatial intervals involves the possibility of discovering or placing some objects among those given.

Time is reflected not in the reflection of the ordering of objects but in the reflection of the ordering of their changes. "Marks" of space are empirical objects; "marks" of time are changes. This fact is decisive in any discussion of problems relative to the concept of time.

Every change takes place, of course, in time. But there are changes such that in fixing them the reflecting subject (man or instrument) does not take into consideration that they require time. One considers only that these changes take place. These changes are moments of time ("marks" of time). The interval of time, during which these changes happen is in fact greater than zero. But one acts as if it is equal to zero since the reflection of time has to begin with something.

Moments of time are changes such as are important for the reflecting subject only from the following point of view: 1) their order is important;

2) their number is important (for the measure of time). Thus the moment is a limit of the temporal interval.

When some changes are selected as such "marks" of time, then the temporal interval is fixed as follows: 1) one observes one change and then one observes another change; 2) the interval between them relative to some change, taken as point of reference, is the temporal interval. Thus, for some changes their order is the result of observation (the sequence of observations coincides with their objective sequence), i.e., the temporal relations of the order are given. For other changes it is the case that they are defined through the others.

We said above that in the fixing of changes the investigator observes the states of objects in temporal sequence (one after the other). But here we say that the fixing of changes is a means of fixing time. There is no circle here. In the first case we do not assume that the investigator has a concept of time. In the second, however, we are talking about the reflection of time in the terminology of the investigator.

The time of object s is the interval of time

$$\langle s \urcorner \leftarrow E \Rightarrow s \leftarrow E, \quad s \leftarrow E \Rightarrow s \urcorner \leftarrow E \parallel \delta \rangle .$$

The temporal relations of objects are the relations of their times (i.e., only the indicated intervals).

The unit of measure of time is a temporal interval. The magnitude of a temporal interval is ultimately determined by the possibility or impossibility of certain changes between its limiting moments.

Space and time are not empirical objects. It is impossible singly to observe empirical objects and their changes, on the one hand, and space and time, on the other. It is meaningless to talk about changes, the coming-to-be and the passing away of space and time. It is meaningless to talk about the speed of time. The expressions "space changes", "properties of space", "the structure of space", "changes of time", "the speed of time", "the flow of time", "different direction of time", "reverse flow of time", etc., which are often found in pseudoscientific and scientific works, are either meaningless and inconsistent or have meaning only as literary paraphrases of terms which designate spatial and temporal orders of objects, spatial and temporal intervals, series, structures, etc. All the tricks with the concepts of space and time which have excited the imagination of readers over the years are based on the lack of clarity and precision of the usual expressions and on their confused comprehension. For example, one only has to take space as a set of objects, included in some... spatial (in the above sense) structure in some... time in order to produce a series of assertions in conflict with "common sense" and the usual notions.

Space and time do not exist in the same sense that empirical objects do. They exist for the investigator only if he is in a position to select at least two different empirical objects forming a spatial structure for him and (for time) at least two different changes forming a temporal structure for him.

Thus, in the making precise of a series of concepts having to do with space and time we can use the terminology of the logic of relations: in so doing we regard spatial and temporal relations as special cases of relations in general, applying to them the corresponding assertions of logic.

The physicist, for example, could assert: "The world has no beginning in time", "The world has a beginning in time", "Time is continuous", "Time is discontinuous", "The world does not have a beginning in space", "The world is extended in space", etc. All these assertions are extra-logical. But they are semiotic structures, the properties of which are described in logic. And if we explicate all their signs we get a series of assertions.

$$(\neg\exists s^1)(\forall s^2)((s^2 > s^1 \mid \delta):(s^2 = s^1 \mid \delta))$$
$$(\exists s^1)(\forall s^2)((s^2 > s^1 \mid \delta):(s^2 = s^1 \mid \delta))$$
$$(\neg M s^1)(\forall s^2)((s^2 > s^1 \mid \delta):(s^2 = s^1 \mid \delta))$$

etc., the logical structure of which is clearly expressed. Logic cannot tell physics to accept or reject a given assertion. But it is fully competent to indicate the limits, beyond which science cannot go in setting up hypotheses; it formulates a sort of logical "taboo". And these limits are *a priori*, i.e., they result from previous definitions and not from experience just as we can correctly speak *a priori* about the impossibility of the event $x \cdot \sim x$.

Let us take an example. Let state x be "Body s is in space u^1", and $\neg x$ "Body s is not in space u^1". Let state y be "Body s is not in space u^2", and $\neg y$ "Body s is in u^2". Let s be changed from u^1 to u^2 and the latter will not coincide, i.e., $(\exists\delta^*)(u^1 > u^2 \mid \delta^*)$. In order to formulate the transfer, the investigator has to establish $x \Rightarrow \neg x$ and then $y \Rightarrow \neg y$. In this way we obtain the interval of time $\langle x \Rightarrow \neg x, y \Rightarrow \neg y \mid \delta \rangle$, which is greater than zero (no matter how fast s moves), since $((y \Rightarrow \neg y) > (x \Rightarrow \neg x) \mid \delta)$. Thus, from the point of view of the investigator who is observing the transfer of the body from one region of space to the other, the assertion that this transfer cannot be instantaneous (without passage of time) is *a priori*: it flows from the very concepts being used. If the investigator assume the possibility of instantaneous transfer he is assuming some abstract objects (which is equivalent to not paying attention to the time necessary for the transfer).

Another example. It is clear that the spatial interval between the empirical objects s^1 and s^2 can change: there are cases where $\langle s^1, s^2 \mid \delta \rangle = \alpha$ in time r^1 and $\langle s^1, s^2 \mid \delta \rangle = \beta$ in time r^2 but $\alpha \neq \beta$. Moreover we here have in mind the change of interval between the very same objects (the latter are preserved). Changes of objects happen in such a way that they do not exist in the same sense as empirical objects. They are not preserved. If a and b are any changes and for a judgement on change $\langle a, b \mid \delta \rangle$ one has to observe $\langle a, b \mid \delta \rangle = \alpha$ in one time and $\langle a, b \mid \delta \rangle = \beta$ in another, where $\alpha \neq \beta$, then this is excluded by the very meaning of the concepts. The only thing which is possible here is a difference of intervals between different representatives of Ka and Kb: $\langle a^1, b^1 \mid \delta \rangle \neq \langle a^2, b^2 \mid \delta \rangle$, where $a^1 \in Ka$, $a^2 \in Ka$, $b^1 \in Kb$, $b^2 \in Kb$. All that has been said applies to temporal relations of objects in general.

The division of labor in science makes it so that problems of spatial-temporal properties are left to physics. Physics develops a corresponding theory of measurement, dealing with the speed of signals on the occurrence of events, the transfer of events and observers, interchange of events, etc. Here certain difficulties arise; paradoxical situations take place. But these difficulties and "paradoxes" are only indications of the complexity of the problem of measuring spatial-temporal relations. They have an extra-logical character. And when one tries to consider them as inconsistent with an outdated logic, one only confuses something which is relatively simple from a logical point of view.

Let the symbols

$$a^1, a^2, a^3, \ldots$$

be the terms of spatial structures and

$$b^1, b^2, b^3, \ldots -$$

of temporal ones. The symbols a and b are "domain (structure) of space" and "temporal interval", respectively. The symbols

$$s \leftarrow a^i \quad \text{and} \quad s \leftarrow b^i$$

will represent the sentences "s is found in a^i" and "s exists in b^i".

Science uses the sentences

$$(\alpha \succ\!| a)((s \downarrow a)\, \beta \leftarrow P), (\alpha \succ\!| b)((s \downarrow b)\, \beta \leftarrow P)$$
$$(\alpha \succ\!| a)((s\beta \leftarrow a), (\alpha \succ\!| b)(s\beta \leftarrow b)),$$

where the quantifier binds only the terms of time and space. Usually these quantifiers are the words "everywhere", "sometimes", "never", "always", etc. The term s can be individual.

Let sv and st indicate, respectively, "Individual s in region of space v" and "Individual s at time t". For sv it is clear that $sv \in Ks$. But one can also consider st, i.e., one can accept the axiom $st \in Ks$. Thanks to this, one and the same object can be regarded as a class of two and more objects in time.

4. PART AND WHOLE

Greater precision in a series of ontological concepts leads to consequences which are easily seen in the case of the concepts of "part" and "whole".

One often sees in logical-mathematical works the statement that the assertion "The part is smaller than the whole" is not true in regard to infinite classes: there are cases where the power of a proper subclass is equal to that of the class itself. This statement is devoid of meaning since all the terms ("part", "smaller than" and "whole") are indeterminate and ambiguous in this statement. In fact what can the term "smaller than" mean in reference to classes? It can mean that the power of one class is smaller than that of the other. But it can mean that there is at least one individual which is included in one class but not in the other. And the terms "part" and "whole" can designate relations between classes, between segments and series, between structures and sub-structures, etc. All these terms have to be clarified in order that the assertion "The part is smaller than the whole" have a sense.

But no matter how we try to clarify the concepts "part" and "whole" we have to introduce a series of concepts which make them superfluous.

However, these terms are usually used (on the intuitive level) to designate the relations between segment and series, between structure and substructure. And the term "smaller than" concerns a comparison of their lengths and "extensions".

The concepts of part and whole are not the only general concepts which can be clarified only by the introduction of sets of other concepts: the result of such a clarification is that the concepts in question become purely literary expressions or expressions with very amorphous sense.

5. CAUSE

The concepts "part" and "whole" are not the only ones of the kind just described. Among the others we find the concept "cause".

Science often uses the sentences "*x* is the cause of *y*" and "*x* is the effect of *y*". The fact that the term "cause" (and "effect", derived from it) is not univocal causes numerous and fruitless discussions. But no matter what meaning is in question what does not change is the fact that sentences of this kind are shorthand for some set of sentences.

The sentence "*x* is the cause of *y*" is used in particular as shorthand for the following set of sentences: 1) $\sim X \rightarrow \sim Y$; 2) $\neg x \Rightarrow x$ obtains; 3) then (after this) $\neg y \Rightarrow y$ obtains. In addition, one presupposes some limited spatial-temporal domain where the events and changes are observed. Obviously there is no causal connection between simultaneous (in this limited domain) events.

The sentence "*x* is the cause of *y*" is also used in the sense $(\neg x \Rightarrow x) \rightarrow (Rx)(\neg y \Rightarrow y)$, where Rx is "after this". In the case of individually occurring events this sentence is sometimes used in the sense "If *x* were not, *y* would not be".

One also meets the following understanding of cause: the cause of the event *y* is everything which makes *y* arise (everything which is necessary for the appearance of *y*). Here the real is often confused with the reflected, begetting insoluble contradictions. In fact, for the coming to be of any empirical event an infinite set of other events is really needed and the fixing of the cause of an event is therefore practically impossible. Science, however, deals only with some of the events which precede the events in which it is interested. The rest occur independent of knowledge. The events which are formulated are in a way attributes of the infinite set of events which make it possible to make prognoses relative to the occurrence of other events. And if by "cause" one means such events, fixed in a certain type of sentence, then knowledge of causes of events is a trivial fact of science.

The above mentioned right to fix as cause not the whole set of events which precede the appearance of y but only its "attributes" (i.e., only some of the events, say x), is based on the investigator's succeeding in the selection of an x such that its occurrence presupposes the occurrence of all other events necessary for the appearance of y:

$$\begin{aligned} &X \cdot Z^1 \cdot \ldots \cdot Z^n \rightarrow (Rx)\, Y \\ &X \rightarrow Z^1 \cdot \ldots \cdot Z^n \end{aligned}$$

Sometimes x is called the cause of y if one presupposes that

$$(X \cdot Z \rightarrow (Rz)\, Y) \cdot (\sim X \cdot Z \rightarrow (Rz) \sim Y).$$

Possible, too, are the definitions:

1) $\downarrow X^i$ is a positive cause of $\downarrow Y^i$ if and only if

$$\begin{aligned} &(\forall \downarrow X)(\forall \downarrow Y)(\forall \downarrow v)(X \rightarrow (Rx)\, Y)/v \\ &\downarrow X^i \in K \downarrow X, \quad \downarrow Y^i \in K \downarrow Y, \quad (\downarrow v^i \in K \downarrow v) \\ &X^i \cdot (Rx^i)\, Y^i \cdot v^i \end{aligned}$$

2) $\downarrow X^i$ is a negative cause of $\downarrow Y^i$ if and only if

$$\begin{aligned} &(\forall \downarrow \sim X)(\forall \downarrow \sim Y)(\forall \downarrow v)(\sim X \rightarrow (R \sim x) \sim Y)/v \\ &(\downarrow X^i) \in K(\downarrow X), \quad (\downarrow Y^i) \in K(\downarrow Y), \quad (\downarrow v^i) \in K(\downarrow v) \\ &X^i \cdot (Rx^i)\, Y^i \cdot v^i \end{aligned}$$

In short, when we use the word "cause" we often have in mind situations which are completely different from the logical point of view, and this results in fruitless discussions.

CHAPTER TWENTY-ONE

THE UNIVERSALITY OF LOGIC

1. DOUBTS ABOUT THE UNIVERSALITY OF LOGIC

There is a view that the laws (assertions) of logic are not universal, i.e., that there are cases where one and the same law of logic leads to valid results in one domain of science and to erroneous results in another; as if the laws of logic could admit of exceptions, depending on the object-range. Definite facts are cited in support of this opinion. Already in Hegel's time there was a tradition which rejected the law of contradiction in reference to transitional states of objects. To this are added today views which limit the law of excluded middle and of double negation (in intuitionist and constructivist logic) and also the laws of commutativity and distribution (in "quantum logic").

If logic is really not universal and one for all sciences, then its theses do not have *a priori* force for science and its very use is cast in doubt. Happily for logic this view is due to a misunderstanding.

One has to ask:

1) Why are just these laws of logic considered not universal and not others?

2) Are there cases where other laws of logic are also non-universal?

3) Are there, all the same, laws of logic which are universal?

4) Where is the border line between the universal and non-universal laws of logic?

It is impossible to answer such questions in a non-scholastic way. The laws of logic are assertions which describe (define) the properties of a certain type of sign and of the semiotic structures which contain them. They are assertions about signs and not about the natural objects which are designated by these signs. They provide no information about the objects to which the terms and sentences refer. Even such assertions of logic as "*X* or non-*X*", "It is impossible that there be *X* and non-*X*", etc., are accepted not because such are the objects reflected in *X* but because such are the properties of the signs "or", "and", "not". By their very essence the laws of logic are universal, do not admit of exceptions, and do not depend on the characteristics of this or that domain. These characteristics do determine which of the laws of logic will be used there. But this has nothing to do with the notion of non-universality of logic.

As for the facts which supposedly support this contention there is no difficulty in showing that they result from a confusion of different logical forms.

2. EXAMPLES OF THE "NON-UNIVERSALITY" OF LOGIC

Let us take the negation of the law of non-contradiction relative to transitional states of objects (the "paradox of change"). It is obtained as follows: in addition to the states where $s\leftarrow P$ and $s\urcorner\leftarrow P$ ($s\leftarrow E$ and $s\urcorner\leftarrow E$) we assume a third state, relative to which $\sim(s\leftarrow P)$. $\sim(s\urcorner\leftarrow P)$ (correspondingly $\sim(s\leftarrow E)\cdot\sim(s\urcorner\leftarrow E)$) is true. But the notation of the states, the notation of the assumed third state, and the consequent reasoning are effected within the limits of the system of assertions of two-valued logic. One assumes at least a trivalence of the sentences and proceeds according to logical rules which exclude this third state. In fact, the state, where $s\urcorner\leftarrow P$, is written as $\sim(s\leftarrow P)$, because one does not distinguish $\urcorner$ and $\sim$. The assumption relative to the third state is written in the form $\sim(s\leftarrow P)\cdot\sim\sim(s\leftarrow P)$. And since $\sim\sim(s\leftarrow P)\vdash(s\leftarrow P)$, one obtains $\sim(s\leftarrow P)\cdot(s\leftarrow P)$, correspondingly. Thus, it ends up that $\sim X\cdot X$ will be true for some sentences X. But as we have seen this conclusion is the result of an error. $s\urcorner\leftarrow P$ and $s\urcorner\leftarrow E$ do not always coincide with $\sim(s\leftarrow P)$ and $\sim(s\leftarrow E)$, and $\sim(s\urcorner\leftarrow P)\vdash(s\leftarrow P)$ and $\sim(s\urcorner\leftarrow E)\vdash(s\leftarrow E)$ are not true in the general case (because there can be the cases $s?\leftarrow P$ and $s?\leftarrow E$ which formulate, in particular, the transitional states of the objects).

Similarly, in the case of the law of excluded middle and of the elimination of double negation in intuitionist logic. If we assume the possibility of $s?\leftarrow P$ and $s?\leftarrow E$ then in the general case $(s\leftarrow P):(s\urcorner\leftarrow P)$, $(s\leftarrow E):(s\urcorner\leftarrow E)$; $\sim(s\urcorner\leftarrow P)\vdash(s\leftarrow P)$ and $\sim(s\urcorner\leftarrow E)\vdash(s\leftarrow E)$ are not true. Now if we regard $\urcorner$ as $\sim$, then we obtain that $(s\leftarrow P):\sim(s\leftarrow P)$, $(s\leftarrow E):\sim(s\leftarrow E)$, $\sim\sim(s\leftarrow P)\vdash(s\leftarrow P)$ and $\sim\sim(s\leftarrow E)\vdash(s\leftarrow E)$ are not always true, which is erroneous. If we introduce a new form of negation (say $\lrcorner$) combining the properties of $\sim$ and $\urcorner$, then we obtain true assertions on the admissibility of X: $\lrcorner X$ and $\lrcorner\lrcorner X\vdash X$. But this will in no way affect the universality of the assertions $X{:}\sim X$ and $\sim\sim X\vdash X$, which are parts of the definition of the signs $:$ and $\sim$.

The same happens if we confuse falsity with negation, i.e., if we implicitly or explicitly accept $[X]\leftarrow v^4\dashv\vdash\sim X$. Since $([X]\neg\leftarrow v^4)\dashv\vdash\sim([X]\leftarrow v^4)$ and $\sim([X]\leftarrow v^4)\vdash([X]\leftarrow(v^1{:}v^2{:}v^3))$, then $\sim([X]\leftarrow v^4)\vdash([X]\leftarrow v^1)$ is not always true and, therefore, $\sim([X]\leftarrow v^4)\vdash X$. Accepting $([X]\leftarrow v^4)\dashv\vdash\sim X$ we obtain the conclusion that $\sim\sim X\vdash X$, which is not always true. But this is erroneous since $\sim X\vdash([X]\leftarrow v^4)$ is not always true (if $[X]\leftarrow v^2$, then $[\sim X]\leftarrow v^1$ and $[[X]\leftarrow v^4]\leftarrow v^4$). Similarly it is erroneous from the fact that $([X]\leftarrow v^1){:}([X]\leftarrow v^4)$ is not always true to conclude to the negation of the universality of $X{:}\sim X$, because one accepts $\sim X\dashv\vdash([X]\leftarrow v^4)$. There are other variants of this type of reasoning.

The confusion of different logical forms is also behind the view that the laws of commutativity and distribution for conjunction and disjunction ($X\cdot Y\vdash Y\cdot X$, $X\vee Y\vdash Y\vee X$, $(X\vee Y)\cdot Z\vdash X\cdot Z\vee Y\cdot Z$, $X{:}Y\vdash Y{:}X$, etc.) have to be subjected to limitations in the field of quantum mechanics. It is quite possible that for the description of the relations between objects in the domain of quantum mechanics one will have to introduce some logical signs which will partly coincide with the signs $\cdot$, $:$ and $\vee$, but will partly be distinguished from them (for example, for them by definition the laws analogous to those for commutativity and distribution for $\cdot$, $:$, $\vee$ will not be valid). This does not at all mean that the above mentioned laws are not universal. If we agree to introduce a sign $\cdot$ such that $X\cdot Y\vdash Y\cdot X$ then it is stupid to look for cases where this assertion is not true. But if for some sign (say $\cdot\cdot$) we consider $X\cdot\cdot Y\vdash Y\cdot\cdot X$ unacceptable, then this sign is automatically different from the sign $\cdot$. There are no laws of logic which are applicable in the macrocosm and inapplicable in the microcosm. It is possible here only to have logical signs which are suited for the description of objects of the microcosm and different from signs already known in logic. But the assertions which define the properties of these signs will still be universal.

Let us take an example. Let the relations between events x and y be such that: 1) if $\neg x\Rightarrow x$ obtains, then $\neg y\Rightarrow y$ is possible and $x\cdot y$ is possible; 2) but if $\neg y\Rightarrow y$ obtains, then $\neg x\Rightarrow x$ is impossible and $x\cdot y$ is impossible. Examples of this kind are no less frequent in macrocosm than in microcosm. One can introduce a special logical sign for such relations between events and by definition it will not have the property of commutativity. The possibilities of forming such signs are unlimited, provided there are serious reasons for this and the procedure is considered expedient.

3. MANY-VALUED LOGIC AND THE UNIVERSALITY OF LOGIC

Reference to many-valued logic in "arguing" the thesis on the non-universality of logic is devoid of any meaning. One can construct a many-valued logic where some tautologies ("laws") of two-valued logic will not be tautologies (in the three-valued systems of Łukasiewicz or Heyting, for example). But such a many-valued system can be devised for any tautology of two-valued logic. What is more, even the expression "this tautology of two-valued logic is not a tautology in this many-valued logic" has to be clarified.

We adopt the following definitions:

D1. Let $F^n(p^1,\ldots,p^m)$ (where $n \geqslant 1$, $m \geqslant 1$) be a function of some many-valued logic. If in the corresponding matrix (truth-table) one strikes all truth-values except two which correspond to the values of two-valued logic and the resultant matrix (table) will be that of some function $F^2(p^1,\ldots,p^m)$ of two-valued logic, then F^n will be called the many-valued analogon of F^2, and F^2 the two-valued analogon of F^n (the functions F^2 and F^n are analogous functions).

D2. A sentential formula of two-valued (many-valued) logic is the analogon of a sentential formula of many-valued (two-valued) logic if and only if one of them can be obtained from the other by replacing the signs of functions with the corresponding signs of analogous functions.

Let PL^2 be a functionally complete system of two-valued sentential logic and PL^n a many-valued system. It is trivially easy to prove the following assertions

T1. For any tautology α^2 in PL^2 one can construct a PL^n such that α^n, analogous to α^2, will not be a tautology in it.

T2. If PL^n is functionally complete then it is possible to find at least two different analoga F_1^n and F_2^n for every F^2 in it; for every sentential formula α^2 in PL^2 one can find at least two different formulae α_1^n and α_2^n in PL^n, analogous to it;

T3. For any tautology α^2 in PL^2 in the functionally complete system PL^n one can find (define) such analoga of the functions occurring in α^2

that the α_1^n analogous to it is a tautology in PL^n and α_2^n is not.

We adduce the proof of *T3*. All the tautologies of PL^2 are equivalent; all the functions of PL^2 are defined by means of $\vee$ and $\sim$. Therefore, it is enough to take the formula $p \vee \sim p$ and to construct three-valued analoga for $\vee$ and $\sim$. Let the truth-values be 1 and 3 in two-valued logic and 1, 2 and 3 in three-valued logic. In both cases tautologies will always take the value 1. In two-valued logic $\vee$ and $\sim$ are defined as follows: 1) $p \vee q = min\,(p, q)$; 2) if $p=1$, then $\sim p=3$; if $p=3$, then $\sim p=1$. In three-valued logic the definition of $\vee$ remains the same, so that the three-valued disjunction is obviously an analogon of the two-valued. For negation there are two possible three-valued analoga. The first is obtained by expanding the definition of 2 with the point: a) if $p=2$, then $\sim p=2$. The second analogon is obtained by adding to the definition of 2 the point: b) if $p=2$, then $\sim p=1$. Now it is easy to see that the three-valued $p \vee \sim p$ will be a tautology if the three-valued $\sim$ is the second analogon of the two-valued and will not be a tautology if the three-valued $\sim$ is the first analogon of the two-valued.

4. DIFFERENCES IN LOGICAL SYSTEMS

The fact of multiplicity of logical systems is not an argument in favor of the thesis on the non-universality of logic. We will leave aside differences in viewpoints, abilities and interests of logicians, differences in interpretation of logical calculi, different trends in logic and in its development, etc. We will remain with the case which interests us most: we have two logical theories which supposedly describe the properties of the same logical signs; however, the sets of formulae provable in them (which means the sets of rules of logic which are assumed) do not coincide. This can mean only one thing: that these systems define different collections of logical signs; they must differ by at least one logical sign.

The classical and intuitionist (constructivist) sentential calculi are cases in point. They both pretend to provide definitions of the properties of the signs "and", "or" and "not". However, in fact they define different negations: not everything that is true for classical negation is true for the intuitionist negation. And it would be wrong to think that there is some kind of natural negation which could be learned with different degrees of depth, completeness and exactitude, just as one learns about atoms, substance, animals, etc., and the properties of which the "intuitionists" would perceive more clearly than the "classicists" (or vice versa). There is progress here. But it consists in the fact that relative to some requirements of knowledge negation has been differentiated and one has constructed logical systems for its different forms, which define its properties.

Differences in logical systems (provided they are not variations of one and the same) indicate the expansion and enrichment of the apparatus of logic; they indicate the appearance of new logical methods (in particular through differentiation, limitation, etc., of what is already on hand). But this does not at all mean that the same laws of logic are true in some domains of science and not-true in others.

It should be noted here that the effort to present classical mathematical logic as the only method for solving all problems of the logical theory of scientific knowledge (i.e., as the only valid conception of logic in general) seems unjustifiable. In many cases its use has had an illusory effect and has generated paradoxical situations and taken us up blind alleys. It will be closer to the truth to say that classical mathematical logic is just one of the methods of the logical theory of scientific knowledge and, with suitable interpretations, is one of its sections. Criticism of the notion of the universality of logic in the directions discussed above has destroyed the idea that classical logic alone is suited for the solution of all problems of the logical theory of scientific knowledge (and is "universal" in this sense). That the elaboration of logic along these lines is the elaboration of new sections of universal logic (in the sense defined above) should not be surprising in view of the history of science.

CONCLUSION

The following errors are to be found in contemporary logical theory of scientific knowledge:

1) the absolutization of classical mathematical logic, i.e., the effort to present it as *the* method for the solution of all problems of the theory of scientific knowledge; if there is a repudiation of the classical conception of logic, then an analogous fate awaits another conception (for example, the intuitionist theory of deduction is regarded as a replacement for the classical theory);

2) triteness in the solution of various problems of the theory of scientific knowledge; the effort to formulate and consider the problems themselves exclusively in a form which makes it possible to use the ordinary methods of logic.

As a result of these errors the investigation of many problems of the theory of scientific knowledge has ended up in a blind alley and has generated "paradoxical" situations; the greater portion of the results are purely illustrative. These results clearly demonstrate the character of the methods of logic but are not sections of some single and systematic construction of a science concerning scientific knowledge.

It is our view that one can construct a theory of scientific knowledge adequate to the practice of contemporary science only with the help of various logical calculi, methods, conceptions, trends, etc. To accomplish this the very problems of logic have to be expanded and reoriented relative to the tradition which has been built up in logic. We have tried to show the possibilities which can be discovered already in the analysis of the most fundamental concepts and principles of logic if they are regarded as means of describing the properties of scientific knowledge.

APPENDIX

(See Chapter Seven)

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PROOF OF THE BASIC THEOREMS OF THE THEORY OF LOGICAL ENTAILMENT

A. A. Zinov'ev has constructed logical systems for the different forms of logical implication: strong, maximal, converse, and weakened. Below, we prove the basic metatheorems of these systems.

Some remarks on the form of the exposition. Within each section the theorems (formulae which are provable in some systems) and metatheorems are numbered separately. Citations of theorems of another section will include the number of the theorem plus that of the section: e.g. T_2^{II} will designate T_2 of section II. In some cases in the proof of theorems we will write, on the right-hand side and in square brackets, the theorems and rules of inference used to obtain the theorem in question.

1. SOME THEOREMS OF S^1

We shall prove a series of theorems which will be needed later.

T_1. $pq \vdash q$ $[A_3, A_2, R_1, R_2]$
T_2. $p \vdash p$ $[A_1, R_2]$
T_3. $p \vdash pp$ $[T_2, R_3]$
T_4. $pp \vdash p$ $[A_2, R_1]$
T_5. $qp \vdash pq$ $[A_3, R_1]$
T_6. $p{:}q \vdash q{:}p$
  1. $\sim(p{:}q) \vdash qp{:}\sim q \sim p$ $[A_6, A_3, T_5, R_4]$
  2. $\sim(p{:}q) \vdash \sim(q{:}p)$ $[1, A_6, R_1, R_2]$
  3. $\sim(q{:}p) \vdash \sim(p{:}q)$ $[2, R_1]$.
  4. $\sim\sim(p{:}q) \vdash \sim\sim(q{:}p)$ $[2, 3, R_4]$
  5. $p{:}q \vdash q{:}p$ $[A_1, 4, R_2]$
T_7. $q{:}p \vdash p{:}q$ $[T_6, R_1]$

T_8. Proof of the commutation rule for $p^1{:}\ldots{:}p^n$ $(n \geqslant 2)$ is based on A_4, A_6, A_7, A_8, T_6, T_7.

In what follows references to A_1, A_4, R_1, T_2–T_8 will in most cases be omitted as evident.

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T_9. $(p:q)\,r \vdash pr:qr$
1. $(p:q)\,r \vdash (q:pr)\,r$ $[A_{10}, T_1, R_3]$
2. $(p:q)\,r \vdash pr:qr$ $[1, A_{10}, R_2]$

T_{10}. For $p^1:\ldots:p^n$ $(n \geqslant 2)$ the theorem is proved as in T_9.

T_{11}. $p \vdash p:p \sim p:p \sim p$ $[T_3, T_4, A_1, A_5, R_1, R_2, R_4]$
T_{12}. $p:p \sim p:p \sim p \vdash p$ $[A_9, T_1, R_2]$
T_{13}. $p \vdash p:p \sim p$
1. $\sim p \vdash \sim(p:p \sim p:p \sim p)$ $[T_{11}, T_{12}, R_4]$
2. $p:p \sim p:p \sim p \dashv\vdash p:(p \sim p:p \sim p)$ $[A_7, A_8]$
3. $\sim p \vdash \sim(p:(p \sim p:p \sim p))$ $[1, 2, R_4, R_2]$
4. $\sim p \vdash p(p \sim p:p \sim p):\sim p \sim (p \sim p:p \sim p)$ $[3, A_6, R_2]$
5. $p \sim p:p \sim p \dashv\vdash \sim(p:\sim p)$ $[A_6]$
6. $\sim p \vdash p(p \sim p:p \sim p):\sim p(p:\sim p)$ $[4, 5, R_4, R_2]$
7. $\sim p \vdash \sim p:p \sim p:p \sim p:p \sim p$ $[6, A_9, T_{10}, A_8, R_4, R_2]$
8. $p \vdash (p:p \sim p:p \sim p):p \sim p$ $[7, R_1, A_7, R_2]$
9. $p \vdash p:p \sim p$ $[8, T_{11}, T_{12}, R_4, R_2]$

T_{14}. $p:p \sim p \vdash p$ $[A_9, T_1, R_2]$
T_{15}. $p:q \vdash p \sim q:\sim pq$
1. $p:q \vdash \sim(pq:\sim p \sim q)$ $[A_1, A_6, R_4, R_2]$
2. $p:q \vdash p \sim pq \sim q:\sim(pq) \sim (\sim p \sim q)$ $[1, A_6, R_2]$
3. $p:q \vdash p \sim q:\sim pq$ $[2, A_5, A_9, T_{10}, A_7, A_8, T_{13}, T_{14}, R_{14}, R_2]$

T_{16}. $p \sim q:\sim pq \vdash p:q$
1. $p \sim q:\sim pq \vdash \sim(p \sim pq \sim q:\sim(p \sim q) \sim (\sim pq))$
$[A_1, A_6, R_2, R_4]$
2. $p \sim q:\sim pq \vdash \sim(pq:\sim p \sim q)$
$[1, A_5, A_9, T_{10}, A_8, A_7, T_{13}, T_{14}, R_4, R_2]$
3. $p \sim q:\sim pq \vdash p:q$ $[2, A_6, A_1, R_4, R_2]$

T_{17}. The theorem for $p^1:p^2:\ldots:p^n$ $(n \geqslant 2)$ is proved as T_{15} and T_{16}.

T_{18}. $p \vdash q(p:\sim p)$ $[T_{13}, A_9, R_2]$
T_{19}. $p(p:\sim p) \vdash p$ $[A_2]$
T_{20}. $pr:q \vdash (pr:q)\,(p:\sim p)$
1. $pr:q \vdash pr \sim q:\sim(pr)\,q$ $[T_{17}]$
2. $pr:q \vdash pr \sim q:\sim prq:p \sim rq:\sim p \sim rq$
$[1, A_5, A_9, T_{10}, R_4, A_8, R_2]$
3. $pr:q \vdash (pr \sim q:\sim prq:p \sim rq:\sim p \sim rq)\,(p:\sim p)$

$[2, T_{18}, T_{19}, R_4, A_9, R_2]$

4. $pr{:}q \vdash p{:}{\sim}p$ $[3, T_1, R_2]$
5. $pr{:}q \vdash (pr{:}q)(p{:}{\sim}p)$ $[4, T_2, R_3]$

T_{21}. $(pr{:}q)(p{:}{\sim}p) \vdash pr{:}q$ $[A_2]$

T_{22}. $(p{:}q)\, p \vdash {\sim}\, q$

1. $(p{:}q)\, p \vdash p{:}pq$ $[T_{10}]$
2. $(p{:}q)\, p \vdash (p{:}pq)(q{:}{\sim}q)$ $[1, T_{20}, R_2]$
3. $(p{:}q)\, p \vdash {\sim}\, q{:}q{:}q$ $[2, A_{10}, A_9, A_2, R_2]$
4. $(p{:}q)\, p \vdash {\sim}\, q$ $[3, T_{17}, A_9, T_1, R_2]$

T_{23}. $(p{:}q) \sim p \vdash q$

1. $(p{:}q) \sim p \vdash p \sim q{:}{\sim}pq$ $[A_2, T_{17}, R_2]$
2. $(p{:}q) \sim p \vdash {\sim}\, p{:}{\sim}q$ $[1, T_{17}, R_1, R_2]$
3. $(p{:}q) \sim p \vdash ({\sim}\, p{:}{\sim}q) \sim p$ $[2, T_1, R_3]$
4. $(p{:}q) \sim p \vdash q$ $[3, T_{22}, R_2]$

The proof of T_{15}, T_{16}, T_{17}, T_{22} and T_{23} is the proof of formulae which are included as axioms of the prime variant of S^1.

2. THEOREMS OF "NON-PARADOXICALITY"

The following metatheorem holds relative to S^1:

T_1^*. If $X \vdash Y$ is a theorem of S^1, then only those (but not necessarily all) variables which occur in X occur in Y.

Proof of T_1^*. First case: $X \vdash Y$ is an axiom of the system S^1. It is easy to see that only those variables which occur in X occur in Y.

Second case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$ and is obtained from $X^1 \vdash X^2$ in accordance with the rule R_1 (substitution). It is clear that if only the variables which occur in X^1 occur in X^2, then only the variables which occur in Y^1 occur in Y^2.

Third case: $X \vdash Y$ has the form $X^1 \vdash Z^1$ and is obtained from $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ by use of R_2. We assume that in the Z^1 of $X^1 \vdash Z^1$ there occurs a variable p which does not occur in X^1. Let it be the case that in the Y^1 of $X^1 \vdash Y^1$ there occur only those variables which occur in X^1. Then in the Z^1 of $Y^1 \vdash Z^1$ there will be the occurrence of the variable p which does not occur in Y^1. Therefore, if in the premisses $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ the Y^1 contain occurrences of only those variables which occur in X^1 and Z^1 contains occurrences of only those variables which occur in Y^1, then in the conclusion $X^1 \vdash Z^1$ only those variables will occur in Z^1, which occur in X^1.

Fourth case: $X \vdash Y$ has the form $X^1 \vdash Y^1 Z^1$ and is obtained from $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ through application of R_3. We presuppose that some variable p occurs in the $Y^1 Z^1$ of $X^1 \vdash Y^1 Z^1$, which does not occur in X^1. Then p occurs either in Y^1 or in Z^1 or in both Y^1 and Z^1. In any of these cases either the Y^1 or the Z^1 or both Y^1 and Z^1 in the premisses $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ will contain a variable p which does not occur in X^1. Therefore, if in the premisses $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ both Y^1 and Z^1 contain only those variables which occur in X^1, then the $Y^1 Z^1$ in the conclusion $X^1 \vdash Y^1 Z^1$ will contain only those variables which occur in X^1.

Fifth case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$, where Y^2 is obtained from Y^1 by substitution for occurrences (at least one) of X^1 in Y^1 of the sentential formula X^2; namely $X^1 \dashv\vdash X^2$. If in the X^2 of $X^1 \vdash X^2$ there occur only those variables which occur in X^1, and in the X^1 of $X^2 \vdash X^1$ only those which occur in X^2, then the very same variables occur in X^1 and X^2. Therefore, the Y^2 of the conclusion $Y^1 \vdash Y^2$ will contain only those variables which occur in Y^1.

The following metatheorem follows directly from T_1^*:

T_2^*. If $X \vdash Y$ and $Y \vdash X$ are theorems of S^1, then the sets of variables occurring in X and in Y are identical.

It follows from T_1^* and T_2^* that the so-called "paradoxical" formulae like $p \vdash q : \sim q$, $p \sim p \vdash q$, $p \vdash \sim (q \sim q)$, etc., are not provable in S^1. Thus, S^1 excludes the "paradoxes" of material and strict implication.

3. CONSISTENCY

We adopt the interpretation:

1) variables take the values 1 and 0;

2) if X has the value 1, then $\sim X$ has the value 0; if X has the value 0, then $\sim X$ has the value 1;

3) $X \cdot Y$ has the value 1 if and only if both X and Y have the value 1;

4) $X^1 : \ldots : X^n$ has the value 1 if and only if one and only one of X^1, $X^2, \ldots, X^n$ has the value 1.

5) $X \vdash Y$ has the value 0 if and only if X has the value 1 and Y the value 0.

D_1. Tautology has the usual definition, i.e., a formula which takes the value 1 under any values for the variables it contains.

T_3^*. If $X \vdash Y$ is a theorem of S^1, then it is a tautology.

Proof of T_3^*. First case: $X \vdash Y$ is an axiom of S^1. It is easy to see that $X \vdash Y$ is a tautology.

Second case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$ and is obtained from $X^1 \vdash X^2$ according to R_1 (substitution). It is clear that if $X^1 \vdash X^2$ is a tautology, then $Y^1 \vdash Y^2$ is also a tautology.

Third case: $X \vdash Y$ has the form $X^1 \vdash Z^1$ and is obtained from $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ through use of R_2. The conclusion $X^1 \vdash Z^1$ takes the value 0 only if X^1 has the value 1 and Z^1 the value 0. If the premisses $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ are tautologies then in all cases where X^1 has the value 1, Y^1 and Z^1 have the value 1. Then the conclusion $X^1 \vdash Z^1$ is also a tautology.

Fourth case: $X \vdash Y$ has the form $X^1 \vdash Y^1 Z^1$ and is obtained from $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ by use of R_3. If the premisses $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ are tautologies, then Y^1 and Z^1 take the value 1 in all cases where X^1 has the value 1. Then the conclusion $X^1 \vdash Y^1 Z^1$ is a tautology.

Fifth case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$, where Y^2 is obtained from Y^1 by the substitution of occurrences (at least one) of X^1 in Y^1 by the sentential formula X^2. If $X^1 \vdash X^2$ and $X^2 \vdash X^1$ are tautologies, then X^1 and X^2 have identical truth-values for the same combination of truth-values of the variables occurring in them. This means that the Y^2 of the conclusion $Y^1 \vdash Y^2$ will take the same truth-values as Y^1 for the same combination of truth-values of the variables occurring in them, so that the conclusion will also be a tautology.

From the metatheorem T_3^* directly follows:

T_4^*. If $X \vdash Y$ and $Y \vdash X$ are theorems of S^1, then X and Y are equivalent (have one and the same truth-value for the same combination of truth-values of the variables occurring in them).

Now we can prove the consistency (non-contradictoriness) of S^1 in the following sense:

T_5^*. If $X \vdash Y$ is a theorem of S^1 and $\sim X$ is not a tautology, then $X \vdash \sim Y$ is not a theorem of S^1.

Proof of T_5^*. Since $X \vdash Y$ is a theorem of S^1 then, on the basis of T_3^* $X \vdash Y$ is a tautology. And, since $\sim X$ is not a tautology, the sentential formula X takes the value 1 for at least one combination of the truth-values of the variables occurring in it. It follows from the interpretation of the sign $\vdash$ that for the same combination of the truth-values of the variables the sentential formula Y takes the value 1. Then for the same combination of truth-values of variables the formula $\sim Y$ has the value 0. Therefore, for this combination of truth-values of the variables occurring in X the formula $X \vdash \sim Y$ has the value 0. As a result, $X \vdash \sim Y$ is not a tautology. Therefore, on the basis of T_3^*, $X \vdash \sim Y$ is not a theorem.

Thus, by virtue of T_1^* and T_3^* we obtain: if $X \vdash Y$ is a theorem of S^1, then $X \supset Y$ is a tautology of two-valued logic such that only those variables which occur in X occur in Y ($\supset$ is material implication).

4. COMPLETENESS

D_2. We introduce the concept of strong tautology: the formula $X \vdash Y$ is a strong tautology if and only if it is a tautology (in the sense of D_1) and Y contains only those variables which occur in X.

The system S^1 is complete in the sense of the following metatheorem:

T_6^*. If $X \vdash Y$ is a strong tautology, then it is a theorem of S^1.

Before proving T_6^* a series of theorems has to be proved.

T_{24}. $p \sim p{:}pq \vdash pq$

1. $p \sim p{:}pq \vdash q$ $[A_9, T_{23}, R_2]$
2. $p \sim p{:}pq \vdash p$ $[A_9, T_1, R_2]$
3. $p \sim p{:}pq \vdash pq$ $[1, 2, R_3]$

T_{25}. $pq \vdash p \sim p{:}pq$

1. $pq \vdash pq(p{:}\sim p)$ $[T_{18}, T_{19}, R_4]$
2. $pq \vdash p \sim p{:}pq$ $[1, T_{10}, A_9, A_2, R_2]$

T_{26}. $p \sim pr{:}q \vdash q$

1. $p \sim pr{:}q \vdash (p \sim pr{:}q)(p{:}\sim p)(r{:}\sim r)$ $[T_{20}, R_3]$
2. $p \sim pr{:}q \vdash (q{:}p \sim pr)(pr{:}p \sim r{:}\sim pr{:}\sim p \sim r)$ $[1, A_9, T_{10}, R_4, R_2]$
3. $p \sim pr{:}q \vdash (prq{:}p \sim pr){:}p \sim rq{:}\sim prq{:}\sim p \sim rq$ $[2, A_{10}, A_7, R_2]$
4. $p \sim pr{:}q \vdash prq{:}p \sim rq{:}\sim prq{:}\sim p \sim rq$ $[3, A_9, T_{10}, R_4, T_{24}, T_{25}, R_2]$
5. $p \sim pr{:}q \vdash q$ $[4, A_9, T_1, R_2]$

We introduce some definitions and prove a series of lemmas, necessary for the proof of T_6^*.

D_3. Definition of a completely disjunctive normal form (cdnf) of a sentential formula:

1) $X{:}\sim X$ occurs in cdnf if and only if X is a sentential variable;

2) $X^1{:}X^2{:}\ldots{:}X^n$ ($n \geqslant 1$) occurs in cdnf if the following conditions are met: a) $X^1, X^2, \ldots, X^n$ are formulae of the type $\alpha^{i1}p^{i1}\cdot\ldots\cdot\alpha^{im}p^{im}$ ($m \geqslant 1$), where $\alpha^{i1}, \ldots, \alpha^{im}$ indicate the presence or absence of the sign $\sim$; b) all $\alpha^{ij}p^{ij}$ are pairwise disjoint and ordered so that if in $\alpha^{ik}p^{ik}$ and $\alpha^{il}p^{il}$ the variable p^{ik} alphabetically precedes p^{il}, then $k<l$; if in the formulae $\alpha^{ik}p^{ik}$ and $\alpha^{il}p^{il}$ the variables p^{ik} and p^{il} are identical, α^{ik} indicates the absence of $\sim$ and α^{il} the presence thereof, then $k<l$; c) all X^i are pairwise disjoint;

3) a sentential formula occurs in cdnf only in function of 1) and 2).

D_4. The formula X^* is a cdnf sentential formula of X if and only if X^* occurs in cdnf (in the sense of D_3) and $X \dashv\vdash X^*$ is a theorem of S^1.

Lemma L_1: for every sentential formula X one can find a cdnf X^*.

Proof of L_1. First case: X is identical to X^*. According to T_2 we obtain $X \dashv\vdash X^*$.

Second case: the sign : does not occur in X and the sign $\sim$ occurs only before sentential variables. According to T_3 and T_4 we obtain $X \dashv\vdash X^*$.

Third case: X has the form $X^1{:}X^2{:}\ldots{:}X^n$ ($n \geqslant 2$). If the first case does not happen, then according to T_{17}, A_5, A_9, T_{10}, T_{13} and T_{14} we obtain $X \dashv\vdash X^*$.

Fourth case: X has the form $\sim(X^1{:}X^2{:}\ldots{:}X^n)$ ($n \geqslant 2$). On the basis of A_6, A_5, A_9, T_{10}, T_{13} and T_{14} we obtain $X \dashv\vdash X^*$.

Fifth case: X has the form $Y \cdot Z$. If the first and second cases do not happen, then according to A_5, A_6, A_9, T_{10} we obtain $X \dashv\vdash X'$ where X' is a formula of the form $X^1{:}X^2{:}\ldots{:}X^n$ ($n \geqslant 2$). According to the third case, we have $X' \dashv\vdash X^*$; whence we get $X \dashv\vdash X^*$ on the basis of R_2.

Sixth case: X has the form $\sim(Y \cdot Z)$. We obtain $X \dashv\vdash X^*$ according to A_5 and so on as in the third case.

D_5. Definition of the cdnf formula of strong logical implication: $X \vdash Y$ occurs in cdnf if and only if the sentential formulae X and Y occur in cdnf, the sets of variables occurring in X and Y are identical, and cdnf Y contains all the formulae of the type $p^j \sim p^j$ which occur in cdnf X.

D_6. The formula $X^* \vdash Y^{**}$ is a cdnf formula of strong logical implication $X \vdash Y$ if and only if the formula X^* is a cdnf sentential formula X having the form $X^1{:}X^2{:}\ldots{:}X^n$ ($n \geqslant 1$) and Y^{**} is a cdnf sentential formula Y^*Z^X $(Z{:}{\sim}Z)$, where Y^* is cdnf Y, Z^X is a formula of the type $Z^{X1}\cdot Z^{X2}\cdot\ldots\cdot Z^{Xm}$ $(m\geqslant 0)$ (where Z^{Xj} is a formula of the type $p^j{\sim}p^j$, occurring in X^* but not in Y^*), and Z is the conjunction of the variables $Z^1\cdot Z^2\cdot\ldots\cdot Z^1$ which do not occur in Y^* but do occur in X^*, with the exception of the variables which occur in X^i and their negations.

Lemma L_2: If $X\vdash Y$ is a strong tautology, then one can find a cdnf $X^*\vdash Y^{**}$ for it. And $X^*\vdash Y^{**}$ is also a strong tautology.

Proof of L_2. In function of L_1, for any formula $X\vdash Y$ one can find a formula $X^*\vdash Y^*$, where X^* and Y^* are the respective cdnf sentential formulae X and Y. It follows from D_4, T_2^* and T_4^* that the formulae X and X^*, respectively, are equivalent and the sets of variables they contain are identical. Thus, if $X\vdash Y$ is a strong tautology, then $X^*\vdash Y^*$ is also a strong tautology. Since Y^* contains only those variables which occur in X^*, then one can find a sentential formula Y^*Z^X $(Z{:}{\sim}Z)$, meeting the conditions of D_6, such that the set of variables occurring in it is identical with the set of variables occurring in X. One can find a cdnf Y^{**} for $Y^*Z^X(Z{:}{\sim}Z)$ on the basis of L_1. If ${\sim}X$ is not a tautology, then Y^{**} is the cdnf of $Y^*(Z{:}{\sim}Z)$. It is clear that in this case Y^{**} is equivalent to Y^*. Consequently, $X^*\vdash Y^{**}$ is in this case a strong tautology. If ${\sim}X$ is a tautology then X^* takes the value 0 in the formula $X^*\vdash Y^{**}$, for any combinations of the truth-values of the variables occurring in it. Therefore, in this case, too, $X^*\vdash Y^{**}$ is a strong tautology.

Lemma L_3: the formula $X\vdash Y$ is a theorem if and only if its cdnf $X^*\vdash Y^{**}$ is a theorem of S^1.

Proof of L_3. Let $X\vdash Y$ be a theorem of S^1. We will show that in this case $X^*\vdash Y^{**}$ is also a theorem of S^1. The formula $X^*\vdash Y^*$, where X^* and Y^* are cdnf of the sentential formulae X and Y, is a theorem of S^1 since it can be obtained from $X\vdash Y$ according to R_2 on the basis of D_3 and L_1. If the set of variables occurring in X and Y are identical and in X^* there are no occurrences of the type $p^j{\sim}p^j$ which are not in Y^*, then $X^*\vdash Y^{**}$ is identical with $X^*\vdash Y^*$. If X^* contains occurrences of formulae of the type $Z^{X1}, Z^{X2},\ldots, Z^{Xm}$ $(m\geqslant 1)$ which are not in Y^*, then on the basis of A_9, T_1, R_2, R_3, we obtain $X^*\vdash Y^*Z^X$, where Z^X is a formula of the type $Z^{X1}\cdot Z^{X2}\cdot\ldots\cdot Z^{Xm}(m\geqslant 1)$. If X^* contains variables $Z^1, Z^2,\ldots, Z^l$ $(l\geqslant 1)$ which do not occur in Y^*, then according to T_{20}, using A_7 and A_8 with the help of R_2 and R_3, we obtain $X^*\vdash Y^*(Z{:}{\sim}Z)$, where Z is a formula of the type $Z^1\cdot Z^2\cdot\ldots\cdot Z^l$ $(l\geqslant 1)$. We have thus shown how to move from $X \vdash Y$ to $X^* \vdash Y^* Z^X(Z : \sim Z)$. According to T_{10} and R_2 then follows $X^* \vdash Y^{**}$.

Let $X^* \vdash Y^{**}$ be a theorem of S^1. We will show that $X \vdash Y$ is then also a theorem of S^1. In fact, on the basis of A_9, T_{10}, A_2, T_1, R_2 and R_4 one can proceed from the formula $X^* \vdash Y^{**}$ to the formula $X^* \vdash Y^*$ and then obtain $X \vdash Y$ on the basis of D_3, L_1 and R_2.

On the basis of L_2 and L_3 the proof of T_6^* can be reduced to that of the following metatheorem:

T_7^*. If the formula $X \vdash Y$ is a strong tautology – $X \vdash Y$ occurring in cdnf – then it is a theorem of S^1.

As a preliminary we will prove the following lemma:

L. Let the formula $X \vdash Y$ be a strong tautology – $X \vdash Y$ occurring in cdnf – with the form $X^1 : X^2 : \ldots : X^k \vdash Y^1 : Y^2 : \ldots : Y^n$. If $\sim X$ is not a tautology then $X^1, X^2, \ldots, X^k$ occur in Y in such a way that they are identical with the formulae (not necessarily all) from $Y^1, Y^2, \ldots, Y^n$ so that $n \geqslant k$.

Proof of *L*. Since $\sim X$ is not a tautology, then it follows from D_2 that for any combination of the truth-values of the variables occurring in X either all X^i take the value 0, or one and only one of X^i takes the value 1 and the rest take 0. Each of X^i takes the value 1 for one and only one combination of truth-values for the variables occurring in X. The same is the case for $Y^1, Y^2, \ldots, Y^n$, too. From D_2 and D_4 it follows that the sets of variables occurring in X^i and Y^j are identical. On the basis of D_2 it further follows that $X^1, X^2, \ldots, X^k$ have to be identical with the formulae from $Y^1, Y^2, \ldots, Y^n$. In fact, if X^i is not identical with any of the formulae $Y^1, Y^2, \ldots, Y^n$, then for some combination of truth-values of the variables occurring in X X^i has the value 1 and Y the value 0. Therefore, if $X \vdash Y$ is a tautology, then $X^1, X^2, \ldots, X^k$ occur in Y so that $n \geqslant k$.

Proof of T_7^*. Let it be the case that in $X \vdash Y$ there is no occurrence of a formula of the type $p^j \sim p^j$. Then $X \vdash Y$ has the form $X^1 : X^2 : \ldots : X^k \vdash Y^1 : Y^2 : \ldots : Y^n$ ($1 \leqslant k \leqslant 2^r$, where r is the number of variables occurring in $X \vdash Y$). In function of L and on the basis of the law of commutation for disjunction T_8, it is enough to show that the formula $X^1 : X^2 : \ldots : X^k \vdash X^1 : X^2 : \ldots : X^n$ ($k \leqslant n \leqslant 2^r$) is a theorem of S^1. Depending on n, the proof is distributed into four cases.

First case: $n = k$. Then $X \vdash Y$ has the form $X^1 : X^2 : \ldots : X^k \vdash X^1 : X^2 : \ldots : X^k$ and is obtained from T_2 through application of R_1.

Second case: $n = 2^r$. According to T_{20}, using A_7 and A_8, we have

$$X^1{:}X^2{:}\dots{:}X^k \vdash (X^1{:}X^2{:}\dots{:}X^k)\,(X^1{:}{\sim}X^1)$$

Whence, according to T_1 and R_2, we obtain:

$$X^1{:}X^2{:}\dots{:}X^k \vdash X^1{:}{\sim}X^1.$$

According to A_5 the negation of a conjunction containing r different variables results in a cdnf made up of 2^r-1 members. Consequently, on the basis of R_4 and A_8, we obtain:

$$X^1{:}{\sim}X^1 \vdash X^1{:}X^2{:}\dots{:}X^n \quad (n=2^r)$$

Whence, according to R_2, we have:

$$X^1{:}X^2{:}\dots{:}X^k \vdash X^1{:}X^2{:}\dots{:}X^n \quad (n=2^r)$$

Third case: $n=2^r-1$. According to the second case, we have:

$$X^1{:}X^2{:}X^k \vdash X^1{:}X^2{:}\dots{:}X^n \quad (n=2^r)$$

Using T_8, A_7, A_5 and R_4, we obtain:

$$X^1{:}X^2{:}\dots{:}X^n \vdash X^l{:}{\sim}X^l \quad (k<l\leqslant 2^r)$$

Whence, according to R_2, it follows that

$$X^1{:}X^2{:}\dots{:}X^k \vdash X^l{:}{\sim}X^l \quad (k<l\leqslant 2^r)$$

Applying R_2 to the formula obtained and to the formula proved in the first case, we obtain:

$$X^1{:}X^2{:}\dots{:}X^k \vdash (X^l{:}{\sim}X^l)\,(X^1{:}X^2{:}\dots{:}X^k)$$

On the basis of A_{10}, we obtain:

$$(X^l{:}{\sim}X^l)\,(X^1{:}X^2{:}\dots{:}X^k) \vdash X^lX^1{:}X^lX^2{:}\dots{:}X^lX^k{:}{\sim}X^l$$

Whence,

$$X^1{:}X^2{:}\dots{:}X^k \vdash X^lX^1{:}X^lX^2{:}\dots{:}X^lX^k{:}{\sim}X^l$$

follows according to R_2. Since X^1 does not occur in $X^1{:}X^2{:}\dots{:}X^k$ and all members of cdnf are different, then the formula X^1X^j $(1\leqslant j\leqslant k)$ contains a variable with its negation. Using A_7 and applying k times T_{26}, we obtain:

$$X^lX^1{:}X^lX^2{:}\dots{:}X^lX^k{:}{\sim}X^l \vdash {\sim}X^l$$

According to R_2, we have:

$$X^1{:}X^2{:}\dots{:}X^k \vdash {\sim}X^l$$
$${\sim}X^l \vdash X^1{:}X^2{:}\dots{:}X^k{:}\dots{:}X^{l-1}{:}X^{l+1}{:}\dots{:}X^n \quad (n=2^r-1)$$

follows according to A_5.

Consequently, according to R_2,

$$X^1:X^2:\ldots:X^k \vdash X^1:X^2:\ldots:X^n \quad (n=2^r-1)$$

Fourth case: $k<n<2^r-1$. Let $l=k+1$. According to the third case

$$X^1:X^2:\ldots:X^k \vdash X^1:X^2:\ldots:X^k:X^{k+2}:\ldots:X^n$$

Let $l=k+2$. Then, according to the third case

$$X^1:X^2:\ldots:X^k \vdash \sim X^{k+2}$$

According to R_3, using T_8 at the same time, we obtain:

$$X^1:X^2:\ldots:X^k \vdash (X^{k+2}:X^1:X^2:\ldots:X^k:X^{k+3}:\ldots:X^n) \sim X^{k+2}$$

On the basis of A_{10}, we obtain:

$$(X^{k+2}:X^1:X^2:\ldots:X^n) \sim X^{k+2} \vdash X^{k+2} \sim X^{k+2}:X^1:X^2:$$
$$:\ldots:X^n$$

According to T_{26}, using A_7, we obtain:

$$X^{k+2} \sim X^{k+2}:X^1:X^2:\ldots:X^n \vdash X^1:X^2:\ldots:X^n \quad (n=2^r-2)$$

Applying R_2 to the last three formulae, we have:

$$X^1:X^2:\ldots:X^k \vdash X^1:X^2:\ldots:X^n \quad (n=2^r-2)$$

Proceeding in a similar fashion we can exclude any $X^l (k<l\leqslant 2^r)$ member of disjunction.

Let a formula of the type $p^j \sim p^j$ occur in $X \vdash Y$. Then $X \vdash Y$ has the form $X^1:X^2:\ldots:X^k \vdash Y^1:Y^2:\ldots:Y^n (1\leqslant k\leqslant 2^s, 1\leqslant n\leqslant 2^s$, where s is the number of variables $Z^1, Z^2, \ldots, Z^s$ which occur in $X \vdash Y$ with the exception of p^j). According to T_{20}, T_1, A_7, A_8 and R_2, we have:

$$X^1:X^2:\ldots:X^k \vdash Z:\sim Z$$

where Z is the conjunction of the variables $Z^1 \cdot Z^2 \cdot \ldots \cdot Z^s$.

According to A_9, T_1 and R_2, we obtain

$$X^1:X^2:\ldots:X^k \vdash Z^X$$

where Z^X is the conjunction of all formulae of the type $p^j \sim p^j$.

$$X^1:X^2:\ldots:X^k \vdash Z^X(Z:\sim Z)$$

follows from the obtained formulae according to R_3.

Whence, according to A_5, T_{10}, R_4 and R_2, we obtain:

$$X^1:X^2:\ldots:X^k \vdash Y^1:Y^2:\ldots:Y^n \quad (n=2^S)$$

On the basis of T_8, T_{26}, A_7 and R_2, we obtain the desired formula.

Thus, in function of T_6^* we have: if $X \supset Y$ is a tautology of two-valued logic and Y contains only those sentential variables which occur in X, then $X \vdash Y$ is a theorem of S^1 (this assumes, of course, that X and Y contain only those sentential constants which are in S^1).

5. INDEPENDENCE

Let us agree that if A_i is $X \dashv\vdash Y$, then A_i^1 is $X \vdash Y$ and A_i^2 is $Y \vdash X$. The independence of a series of axioms is established through truth tables with the two truth-values 0 and 1 (the value 1 is the designated value):

1) for A_1^1 we take ${\sim}X=0$ and $X^1:X^2:\ldots:X^n=X^1\cdot X^2\cdot\ldots\cdot X^n$

2) for A_1^2 we take ${\sim}X=1$ and $X^1:X^2:\ldots:X^n=X^1 \vee X^2 \vee \ldots \vee X^n$, where $\vee$ is inclusive disjunction;

3) for A_2 we take ${\sim}X=X$, $XY=1$, $X^1:X^2:\ldots:X^n=1$.

4) for A_3 we take ${\sim}X=X$, $XY=X$, $X^1:X^2:\ldots:X^n=X^1$.

5) for A_5^1 we take ${\sim}X=X$, $X^1:X^2:\ldots:X^n=0$.

6) for A_6^1 we take ${\sim}X=X$, $X^1:X^2:\ldots:X^n=X^1 \vee X^2 \vee \ldots \vee X^n$.

7) for A_6^2 we take $X^1:X^2:\ldots:X^n=X^1 \vee X^2 \vee \ldots \vee X^n$.

8) for A_7 we take $X={\sim}X$, $X^1:X^2=0$, $X^1:X^2:\ldots:X^n=1$, if all $X^i=1$, and $X^1:X^2:\ldots:X^n=0$ in the other cases $(n \geqslant 2)$; $p^1:p^2:\ldots:p^n \vdash (p^1:p^2:\ldots:p^{n-1}):p^n$ is considered a special case.

For proofs of the independence of A_4^1 and A_4^2 one can use three-valued tables with 0 as the only designated value. The following tables are common to A_4^1 and A_4^2: 1) ${\sim}X=1$, if $X=1$; ${\sim}X=2$, if $X=0$; ${\sim}X=0$ if $X=2$; 2) $XY=0$ if and only if $X=0$ and $Y=0$; in the other cases $XY=1$; 3) $X^1:X^2:\ldots:X^n=1$ $(n \geqslant 2)$; 4) $(X \vdash Y)=2$ if $X=0$ and $Y=1$ or $Y=2$; $(X \vdash Y)=0$ in the rest of the cases. Then:

9) for A_4^1 we take $X^1X^2\ldots X^n=0$ $(n>2)$.

10) for A_4^2 we take $X^1X^2\ldots X^n=1$ $(n>2)$.

The independence of A_5^2 and A_9 is established through three-valued truth-tables with 0 and 1 as designated values. Common to their tables is: 1) ${\sim}X=1$ if $X=1$; ${\sim}X=2$ if $X=0$; ${\sim}X=0$ if $X=2$; 2) $(X \vdash Y)=2$

if $X=0$ or $X=1$, and $Y=2$; $(X \vdash Y)=0$ in the other cases. Then:

11) for A_5^2 we take $XY=2$ if and only if $X=2$ or $Y=2$ (or both are the case); $XY=0$ in the other cases; $X^1:X^2:\ldots:X^n=0$ $(n \geqslant 2)$ if one and only one X^i is 0 and all other X^i are 2; $X^1:X^2:\ldots:X^n=2$ if all X^i are equal to 2; $X^1:X^2:\ldots:X^n=1$ in the other cases;

12) for A_9 we take $XY=2$ if and only if $X=2$ or $Y=2$ (or both); in the other cases $XY=1$; $X^1:X^2:\ldots:X^n=1$ $(n \geqslant 2)$.

The independence of A_8 and A_{10} is proved through four-valued tables with 0 as the only designated value. For A_8 there is the special case $Y^1:(Y^2:\ldots:Y^m) \vdash Y^1:Y^2:\ldots:Y^m$, the independence of which is proved on condition that A_7 has the form $p^1:p^2:\ldots:p^n \vdash X$, where X is distinguished from $p^1:\ldots:p^n$ by the order of the brackets with the exception of the cases where p^n is in brackets. Such a formulation of A_7 is enough for the proof of T_8 which can be used to eliminate the case.

Common to A_8 and A_{10} are the tables: 1) $\sim X=X$ if $X=1$ or $X=2$; $\sim X=3$ if $X=0$; $\sim X=0$ if $X=3$; 2) $XY=0$ if and only if $X=0$ and $Y=0$; in the other cases $XY=\mathrm{L}$; 3) $(X \vdash Y)=2$ if and only if $X=0$, and $Y=1$, $Y=2$ or $Y=3$; $(X \vdash Y)=0$ in the other cases. Then:

13) for A_8 we take $X^1:\ldots:X^n=0$ if $X^n=2$, $X^1:X^2:\ldots:X^n=2$ in the other cases;

14) for A_{10} we take $X^1:\ldots:X^n=1$ if at least one X^i is equal to 1; $X^1:\ldots:X^n=0$ in the other cases.

The independence of R_1 is evident. Two-valued tables are enough for the proof of R_2 and R_3. The tables for R_2: $\sim X=0$; $XY=0$; $X^1:\ldots:X^n=0$; $(X \vdash Y)=0$, if and only if $X=1$ and $Y=1$. Here $p \vdash p$ is not a tautology. The tables for R_3: $\sim X=X$; $XY=0$; $X^1:\ldots:X^n=0$; $(X \vdash Y)=0$, if and only if $X=1$ and $Y=0$. Here $p \vdash pp$ is not a tautology.

For the proof of the independence of R_4 we use three-valued tables with 0 as designated value: 1) $\sim X=1$, if $X=1$; $\sim X=2$, if $X=0$; $\sim X=0$ if $X=2$; 2) $XY=0$, if and only if $X=0$ and $Y=0$; $XY=1$ in the other cases; 3) $X^1:X^2:\ldots:X^n=1$; 4) $(X \vdash Y)=2$ if and only if $X=0$, and $Y=1$ or $Y=2$; $(X \vdash Y)=0$ in the other cases. Here $p \vdash p:p\sim p$ is not a tautology.

Thus, we can take it as established that an independent and consistent logical system has been found, which is as close as possible to the intuitive conception of logical entailment (in the sense of T_1^*) and complete relative to its intuitive base (in the sense of T_6^*).

6. MAXIMAL ENTAILMENT

The consistency and independence of systems of maximal, converse and weak entailment follow directly from those of S^1. The proof of the completeness of the systems considered consists in the formulation of a suitable definition of the cdnf of the formula of logical entailment of the respective system, which results in the extension of the proof of completeness of S^1 to the system in question.

The system S^3 of maximal entailment in Zinov'ev's work is constructed from S^1 by putting for the second axiom of the latter the following axiomatic schema:

A_2. $XY \vdash X$,

where Y contains only those sentential variables which occur in X.

It seems to us that this formulation can be weakened by the limitation: in A_2 identical sentential variables occur in X and Y. This does not change the class of formulae proved in S^3 as we shall see below.

The following theorems hold relative to S^3:

T_1^*. If $X \vdash Y$ is provable in S^3, then the sets of sentential variables occurring in X and Y are identical. The proof of T_1^* is trivial: all axioms of S^3 have the indicated property and the rules of inference conserve it.

T_2^*. If $X \vdash Y$ is a tautology and the sets of variables occurring in X and Y are identical, then $X \vdash Y$ is provable in S^3.

(In the definition of tautology the sign $\vdash$ is considered the sign of material implication.)

The proof of T_2^* differs somewhat from the proof of the analogous theorem for S^1.

T_1. $(p{:}q)\,r \vdash pr{:}qr$
1. $(p{:}q)\,r \vdash q{:}pr$ $[A_{10}, T_8^{\mathrm{I}}, R_2]$
2. $(p{:}q)\,r \vdash (p{:}q)\,r$ $[T_2^{\mathrm{I}}]$
3. $(p{:}q)\,r \vdash (q{:}pr)\,r(p{:}q)\,r$ $[1, 2, R_3, T_3, T_4^{\mathrm{I}}, R_4]$
4. $(p{:}q)\,r \vdash (q{:}pr)\,r$ $[3, A_2, R_2]$
5. $(p{:}q)\,r \vdash pr{:}qr$ $[4, A_{10}, T_8^{\mathrm{I}}, R_2]$

T_2. $pr{:}q \vdash (pr{:}q)(p{:}{\sim}p)$
1. $pr{:}q \vdash pr \sim q{:}{\sim}(pr)\,q$ $[T_{17}^{\mathrm{I}}]$
2. $pr{:}q \vdash pr \sim q{:}{\sim}prq{:}p \sim rq{:}{\sim}p \sim rq$
$[1, A_5, A_9, T_{10}^{\mathrm{I}}, R_4, A_8, R_2]$

3. $pr{:}q \vdash (pr \sim q{:}{\sim}prq{:}p \sim rq{:}{\sim}p \sim rq)(p{:}{\sim}p)$
   $[2, T^{\mathrm{I}}_{18}, T^{\mathrm{I}}_{19}, R_4, A_9, R_2]$
4. $pr{:}q \vdash (pr{:}q)(p{:}\sim p)$ $[3, A_7, A_5, T^{\mathrm{I}}_{17}, R_4, R_2]$

Using the S^1 definitions of cdnf for sentential formulae and formulae of logical entailment, we adopt the definition:

D_1. The formula $X^* \vdash Y^{**}$ is a cdnf formula of logical entailment $X \vdash Y$ if and only if the formula X^* is a cdnf of the sentential formula X, and Y^{**} is a cdnf of Y^*Z, where Y^* is the cdnf of Y and Z is a formula of the type $Z^{X1} \cdot Z^{X2} \cdot \ldots \cdot Z^{Xm}$ $(m \geqslant 0)$ (where Z^{Xj} is a formula of the type $p^j \sim p^j$ which occurs in the cdnf of X but not in that of Y).

Relative to S^3 there will be the following lemmas, the proof of which is obtained through obvious modifications of the proof of L^{I}_2 and L^{I}_3.

L_1. If $X \vdash Y$ is a tautology and the sets of variables which occur in X and Y are identical then one can find for it a cdnf $X^* \vdash Y^{**}$. In $X^* \vdash Y^{**}$ the sets of variables occurring in X^* and Y^{**} are also identical.

L_2. The formula $X \vdash Y$ is provable in S^3 if and only if its cdnf is provable in S^3.

In this way the proof of T^*_2 is reduced to that of the theorem:

T^*_3. If $X \vdash Y$ is a tautology, the sets of variables occurring in X and Y are identical, and $X \vdash Y$ occurs in the cdnf, then $X \vdash Y$ is provable in S^3.

The proof of T^*_3 is identical with that of $T^{*\mathrm{I}}_7$, except for those steps in the proof of $T^{*\mathrm{I}}_7$, where one uses T^{I}_{26} which is not provable in S^3. It is easy to see that the use of T^{I}_{26} in proof of the completeness of S^3 can be replaced by the use of theorems T^{I}_{13} and T^{I}_{14}, introduced in S^3, employing A_7, A_8 and R_4.

On the basis of T^*_2 we obtain: if $X \supset Y$ is a tautology of two-valued logic and the sets of variables occurring in X and Y are identical, then $X \vdash Y$ is a theorem of S^3.

7. CONVERSE ENTAILMENT

The system of converse entailment S^4 is obtained from S^3 by addition of the axiom

A_{11}. $\sim p \vdash \sim (pq)$.

It should be noted that all the theorems and metatheorems of S^3 hold for S^4 since S^3 is part of S^4.

We will show that the following metatheorem holds in S^4:

T_1^*. If $X \vdash Y$ is provable in S^4, then X contains only those variables which occur in Y.

Proof of T_1^*. First case: $X \vdash Y$ is an axiom of S^4. It is easy to see that X contains only those variables which occur in Y.

Second case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$ and is obtained from $X^1 \vdash X^2$ according to R_1. It is clear that if X^1 contains only those variables which occur in X^2, then Y^1 contains only those variables which occur in Y^2.

Third case: $X \vdash Y$ has the form $X^1 \vdash Z^1$ and is obtained from $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ through use of R_2. If X^1 contains only those variables which occur in Y^1 and Y^1 contains only those variables which occur in Z^1, then X^1 will also contain only those variables which occur in Z^1.

Fourth case: $X \vdash Y$ has the form $X^1 \vdash Y^1Z^1$ and is obtained from $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ by application of R_3. If in the premisses $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ the X^1 contains only those variables which occur in Y^1 and Z^1, then in the conclusion $X^1 \vdash Y^1Z^1$ the X^1 will contain only those variables which occur in Y^1Z^1.

Fifth case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$, where Y^2 is obtained from Y^1 by putting for occurrences (at least one) of X^1 in Y^1 the sentential formula X^2, where $X^1 \dashv\vdash X^2$. If the X^1 in $X^1 \vdash X^2$ contains only those variables which occur in X^2 and the X^2 in $X^2 \vdash X^1$ contains only those variables which occur in X^1, then the sets of variables which occur in X^1 and in X^2 are identical. Therefore, the Y^1 in the conclusion $Y^1 \vdash Y^2$ contains only those variables which occur in Y^2.

As a result of this metatheorem the "paradoxes" of material and strict implication are excluded in S^4.

The system S^4 is complete in the sense of the following metatheorem:

T_2^*. If $X \vdash Y$ is a tautology and X contains only those variables which occur in Y, then $X \vdash Y$ is provable in S^4.

For the proof of T_2^* we need the theorem

T_1. $p \vdash p(q:\sim q)$

1. $\sim p \vdash \sim pq:pq:\sim p \sim q$ $[A_{11}, A_5, R_2]$
2. $p \vdash pq:\sim p \sim q:p \sim q$ $[1, R_1]$
3. $p \vdash (\sim p \sim q:pq:p \sim q)\, p$ $[2, T_2^{\mathrm{I}}, T_8^{\mathrm{I}}, R_3, R_2]$
4. $p \vdash pq:p \sim q$ $[3, A_{10}, A_7, A_8, A_9, T_{10}^{\mathrm{I}}, T_{13}^{\mathrm{I}}, T_{14}^{\mathrm{I}}, R_4, R_3]$
5. $p \vdash p(q:\sim q)$ $[4, A_9, R_2]$.

We adopt the definition of the cdnf of a formula of logical entailment for S^4.

D_1. The formula $X^{**} \vdash Y^{**}$ is a cdnf formula of the logical entailment $X \vdash Y$ if and only if 1) the formula X^{**} is a cdnf of the sentential formula $X^*(Z{:}\sim Z)$ where X^* is a cdnf of X and Z is a conjunction of variables which occur in Y but not in X; 2) Y^{**} is a cdnf of the sentential formula Y^*Z^X, where Y^* is a cdnf of Y and Z^X is a formula of the type $Z^{X1} \cdot Z^{X2} \cdot \ldots \cdot Z^{Xm}$ $(m \geqslant 0)$ (where Z^{Xj} is a formula of the type $p^j \sim p^j$ which occurs in X^* but not in Y^*).

Lemma L_1. If $X \vdash Y$ is a tautology and X contains only those variables which occur in Y, then one can find it a cdnf $X^{**} \vdash Y^{**}$. Here $X^{**} \vdash Y^{**}$ is a tautology and the sets of variables occurring in X^{**} and Y^{**} are identical.

Proof of L_1. On the basis of L_1^{I} – which also holds for S^4 – one can find for any formula $X \vdash Y$ a formula $X^* \vdash Y^*$, where X^* and Y^* are the respective cdnf of the sentential formulae X and Y. From the definition of a cdnf sentential formula T_1^* and T_4^* it follows that X and X^*, Y and Y^*, respectively, are equivalent and the sets of variables occurring in them are identical. Since X contains only those variables which occur in Y one can find a formula $X^*(Z{:}\sim Z)$, satisfying D_1, such that the set of variables occurring in it is identical with that in Y. On the other hand, one can find a formula Y^*Z^X, satisfying D_1, such that all formulae of the type $p^j \sim p^j$ which occur in X^* will also occur in Y^{**}. One can find cdnf for $X^*(Z{:}\sim Z)$ and Y^*Z^X. It is clear that the sets of variables occurring in X^{**} and Y^{**} are identical. Here, if $\sim X$ is not a tautology, Y^{**} has the form Y^* and the value of $X^{**} \vdash Y^{**}$ is identical to that of $X^* \vdash Y^*$. Consequently, $X^{**} \vdash Y^{**}$ is a tautology in this case. If $\sim X$ is a tautology, then X^{**} takes the value 0 for any combination of its variables. Therefore, in this case, too, $X^{**} \vdash Y^{**}$ is a tautology.

Lemma L_2: the formula $X \vdash Y$ is provable in S^4 if and only if its cdnf $X^{**} \vdash Y^{**}$ is provable in S^4.

Proof of L_2. Let $X \vdash Y$ be provable in S^4. In function of L_1, T_1^*, $T_4^{*\mathrm{I}}$, $X^{**} \vdash Y^{**}$ is a tautology and the sets of variables occurring in X^{**} and Y^{**} are identical. Since S^3 is complete relative to tautologies of the type $X \vdash Y$, where the same variables occur in X and in Y, then $X^{**} \vdash Y^{**}$ is provable in S^4.

Let $X^{**} \vdash Y^{**}$ be provable in S^4. Then, on the basis of D_1 and T_1,

$X^* \vdash Y^{**}$ is provable. Whence, according to A_9 and A_2, we obtain $X^* \vdash Y^*$. On the basis of L_1^{I}, we can pass from this formula to $X \vdash Y$.

The proof of T_2^* is now reduced to that of the metatheorem:

T_3^*. If the formula $X \vdash Y$ is a tautology and X contains only those variables which occur in Y and $X \vdash Y$ occurs in a cdnf, then it is provable in S^4.

The proof of T_3^* is identical with that of $T_3^{*\mathrm{II}}$.

On the basis of T_2^*, we have: if $X \supset Y$ is a tautology of two-valued logic, and X contains only those variables which occur in Y, then $X \vdash Y$ is provable in S^4.

8. WEAK ENTAILMENT

The system S^2 of weak entailment is obtained from S^1 by adding

A_{11}. $\sim p \vdash \sim (pq)$

to $A_1 - A_{10}$, and by limiting the transitivity of R_2 as follows: if $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$ when and only when X, Y and Z contain at least one identical sentential variable.

It is clear that all theorems of S^1 are theorems of S^2.

The following metatheorem holds for S^2.

T_1^*. If $X \vdash Y$ is provable in S^2, then X and Y contain at least one identical sentential variable.

Proof of T_1^*. First case: $X \vdash Y$ is an axiom of S^2. Inspection shows that X and Y contain at least one identical variable.

Second case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$ and is obtained from $X^1 \vdash X^2$ according to R_1. It is clear that if X^1 and X^2 contain at least one identical sentential variable, then Y^1 and Y^2 will also contain at least one identical sentential variable.

Third case: $X \vdash Y$ has the form $X^1 \vdash Z^1$ and is obtained from $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ by use of R_2. Since the rule R_2 is applicable only when $X^1 \vdash Y^1$ and $Y^1 \vdash Z^1$ contain at least one identical sentential variable in X^1, Y^1 and Z^1, then the X^1 and Z^1 of the conclusion $X^1 \vdash Z^1$ contain at least one identical sentential variable.

Fourth case: $X \vdash Y$ has the form $X^1 \vdash Y^1 Z^1$ and is obtained from $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$ according to R_3. If in the X^1 and Y^1 and in the X^1 and Z^1 of the premisses $X^1 \vdash Y^1$ and $X^1 \vdash Z^1$, respectively, there is at least one identical occurrence of a sentential variable, then in the X^1 and $Y^1 Z^1$ of the conclusion $X^1 \vdash Y^1 Z^1$ there will occur at least one identical variable.

Fifth case: $X \vdash Y$ has the form $Y^1 \vdash Y^2$, where Y^2 is obtained from Y^1 by putting for occurrences (at least one) of X^1 in Y^1 the sentential formula X^2, where $X^1 \vdash X^2$ and $X^2 \vdash X^1$. If in X^1 and X^2 there is at least one identical occurrence of a sentential variable, then the Y^1 and Y^2 of the conclusion $Y^1 \vdash Y^2$ contains at least one identical variable.

It follows from the metatheorem T_1^* that the so-called "paradoxes" of material and strict implication are excluded from S^2.

S^2 is complete in the sense of the metatheorem:

T_2^*. If $X \vdash Y$ is a tautology and X and Y contain at least one identical sentential variable, then it is provable in S^2.

For the proof of T_2^*, in addition to the theorems of S^1, we need

T_1. $p \vdash p(q{:}\sim q)$ $[T_1^{\mathrm{III}}]$
T_2. $p(q{:}\sim q) \vdash p$ $[A_2]$.

We introduce the definition of a cdnf formula of weak logical entailment.

D_1. $X^{**} \vdash Y^{**}$ is a cdnf formula of the weak logical entailment $X \vdash Y$, if and only if: 1) X^{**} is a cdnf of the sentential formula $X^*(Z^Y{:}\sim Z^Y)$, where X^* is a cdnf of X and Z^Y is a conjunction of the variables $Z^{Y1} \cdot Z^{Y2} \cdot \ldots \cdot Z^{Yl} (l \geqslant 0)$, which occur in Y but not in X; 2) Y^{**} is a cdnf of the sentential formula $Y^*Z(Z^Y{:}\sim Z^X)$, where Y^* is a cdnf of Y, Z is a formula of the type $Z^1 \cdot Z^2 \cdot \ldots \cdot Z^m$ $(m \geqslant 0)$ (where Z^j is a formula of the type $p^j \sim p^j$ which occurs in X^* but not in Y^*), and Z^X is a conjunction of the variables $Z^{X1} \cdot Z^{X2} \cdot \ldots \cdot Z^{Xr}$ $(r \geqslant 0)$, which occur in X^* but not in Y^*, except for variables occurring in X^* along with their negations.

Lemma L_1. If $X \vdash Y$ is a tautology and X and Y contain at least one identical occurrence of a sentential variable, then one can find for it a cdnf $X^{**} \vdash Y^{**}$. Here $X^{**} \vdash Y^{**}$ is a tautology and the sets of variables occurring in X^{**} and Y^{**} are identical.

Proof of L_1. By virtue of L_1^{I} one can find for any formula $X \vdash Y$ a formula $X^* \vdash Y^*$, where X^* and Y^* are, respectively, the cdnf of the sentential formulae X and Y.

According to D_4^{I}, $T_2^{*\mathrm{I}}$, $T_4^{*\mathrm{I}}$ it follows that the formulae X and X^* and Y and Y^* are, respectively, equivalent, and the sets of their variables are identical. Therefore, if $X \vdash Y$ is a tautology and X and Y contain at least one identical occurrence of a sentential variable, then $X^* \vdash Y^*$ is also a tautology, and in X^* and Y^* at least one identical variable will occur.

And one can find a formula $X^*(Z^Y{:}\sim Z^Y)$, satisfying D_1, such that the set of variables occurring in it is identical with that occurring in Y^{**}. On the other hand, there is always a formula $Y^*Z(Z^X{:}\sim Z^X)$, satisfying D_1, such that the set of variables occurring in it is identical with that occurring in X^{**}. Reduction of these formulae to cdnf changes neither their values nor the sets of variables occurring in them. Consequently, one can find a formula $X^{**}\vdash Y^{**}$ with identical occurrence of variables in X^{**} and Y^{**}. If $\sim X$ is not a tautology, then X^{**} and Y^{**} are respectively equivalent to X^* and Y^* so that $X^{**}\vdash Y^{**}$ is a tautology. If $\sim X$ is a tautology, then X^{**} always takes the value 0; consequently, $X^{**}\vdash Y^{**}$ is a tautology in this case, too.

Lemma L_2. The formula $X\vdash Y$ is provable in S^2 if and only if its cdnf $X^{**}\vdash Y^{**}$ is provable in S^2.

Proof of L_2. Let $X\vdash Y$ be provable in S^2. Then, on the basis of L_1^{I}, $X^*\vdash Y^*$, where X^* and Y^* are the cdnf of X and Y respectively, is also provable in S^2. Using T_1 and T_2 one can obtain the formula $X^*(Z^Y{:}\sim Z^Y)\vdash Y^*(Z^X{:}\sim Z^X)$. On the basis of A_9, A_2, R_3, one can pass from this formula to $X^*(Z^Y{:}\sim Z^Y)\vdash Y^*\,Z\,(Z^X{:}\sim Z^X)$. Application of this formula to cdnf results in $X^{**}\vdash Y^{**}$.

Let $X^{**}\vdash Y^{**}$ be provable in S^2. On the basis of A_9, T_{10}^{I}, T_1, T_2, A_2, R_4 and R_2 one can obtain the formula $X^*\vdash Y^*$. Whence L_1^{I} shows us that $X\vdash Y$ is also provable in S^2.

On the basis of L_1 and L_2 the proof of T_2^* is reduced to that of the metatheorem:

T_3^*. If $X\vdash Y$ is a tautology and X and Y contain at least one identical occurrence of a variable and $X\vdash Y$ occurs in the cdnf, then it is provable in S^2.

The proof of T_3^* is fully identical with the proof of the analogous metatheorem for S^1.

On the basis of T_3^*, we have: if $X\supset Y$ is a tautology of two-valued logic and in X and Y there is at least one identical occurrence of a sentential variable, then $X\vdash Y$ is provable in S^2.

It should be noted that if we remove the limitation on the transitivity rule R_2, we obtain S^5, complete in the following sense: if $X\vdash Y$ is a tautology, then it is provable in S^5. In S^5 all the "paradoxes" of material and strict implication are provable.

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INDEPENDENCE IN THE SYSTEMS OF LOGICAL ENTAILMENT

Solution of the problem of independence for the systems S_1–S_4 can best be approached through solution of the same problem for weak logical entailment (for S_2), since the systems S_1, S_3 and S_4 are fragments of S_2 and the independence of a given axiom (rule of inference) in S_2 implies its independence in S_1, S_3 and S_4.

The axiom A_8 is not independent in S_2 and can be obtained as a theorem.

Proof of A_8:

1) $p \vdash p \ [A_1, A_2, R_3]$
2) $p(p \vee q) \dashv\vdash p \ [1, A_3, A_{12}, R_4]$
3) $p \vee pq \dashv\vdash p \ [2, A_9, A_{10}, R_2]$
4) $pr \vee qr \vdash pr \vee qr \vee r \ [A_{12}, R_1]$
5) $pr \vee qr \vdash r \ [3, 4, R_2, R_3]$
6) $pr(p \vee q) \vee qr(p \vee q) \vdash p \vee q \ [5, R_1]$
7) $pr(p \vee q) \dashv\vdash pr \ [1, A_3, A_{12}, R_4]$
8) $qr(p \vee q) \dashv\vdash qr \ [1, A_3, A_{12}, R_4]$
9) $pr \vee qr \vdash p \vee q \ [6, 7, 8, R_2, R_3]$
10) $pr \vee qr \vdash (p \vee q)\, r \ [5, 9, R_4]$.

The rest of the axioms and rules of S_2 are independent. The independence of the bulk of the axioms is proved by interpretation with the two truth-values 1 and 2 (the designated value is 1). The formula of implication $X \vdash Y$ takes the value 2 only when the value of X is 1 and the value of Y is 2; if no other interpretation is provided for $\cdot$, $\vee$ and $\sim$, the interpretation of the usual two-valued logic is accepted.

1. For proof of the independence of A_1, we take $\sim p = 1$.
2. For A_2 we take $\sim p = 2$ and we will say that the formula $X \vdash Y$ has the value 1 in the cases where it contains at least one of the signs $\cdot$ or $\vee$.
3. For A_3 we take $pq = 1, p \vee q = 1, \ \sim p = p$.
4. For A_4 we take $pq = p, p \vee q = p, \ \sim p = p$.
5. For A_5 we take $p^1 \cdot p^2 \cdot \ldots \cdot p^n = 1$, if $n > 2$.

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6. For A_6 we take $p^1 \cdot p^2 \cdot \ldots \cdot p^n = 2$, if $n > 2$.

7. For A_7 we take it that the sign $\vee$ binds more strongly than $\cdot$.

8. For A_9 we take $p \vee q = p$.

The independence of the axioms A_{10}, A_{11} is proved through truth-tables with three truth-values, i.e., 1, 2, 3 (1 is the designated value).

9. For the proof of the independence of A_{10} we take $pq = 3$, when p or q is equal to 3, and $pq = 1$ in the other cases, $p \vee q = {}_{\min}[p, q]$, $\sim p = 4 - p$, the formula $X \vdash Y$ takes an undesignated value only in the cases when the value of X is equal to 1, and Y is 3, and when the value of X is 2 and that of Y equal to 1 or 3.

10. For A_{11} we take $pq = {}_{\max}[p, q]$, $p \vee q = {}_{\min}[p, q]$, $\sim p = 4 - p$, and the formula $X \vdash Y$ has the value 1 when and only when the value of X is greater than or equal to that of Y.

11. For proof of the independence of A_{12} we assume that the formula $X \vdash Y$ takes an undesignated value only in the case where Y contains a sentential variable which is lacking in X.

12. The independence of R_1 is evident: without this rule the formula $q \vdash \sim\sim q$ would be unprovable.

13. The independence of the rule R_2 is proved in three-valued tables. We take $pq = {}_{\max}[p, q]$, $p \vee q = {}_{\min}[p, q]$, $\sim p = 4 - p$, the formula $X \vdash Y$ has the value 3 when the value of X is 1 or 2, and the value of Y is 3, and the value 1 in the other cases. Here, the theorem $p \vee \sim qq \vdash p(q \vee \sim q)$ has the value 3 when $p = 3$ and $q = 2$.

14. For the proof of the independence of R_3 two-valued tables are enough. We take $pq = 1$, $p \vee q = 1$, $\sim p = 1$, $X \vdash Y$ has the value 2 only when the values of X and Y are 2. Here the theorem $p \vdash p$ has the value 2 when $p = 2$.

15. The independence of R_4 is proved with the help of the three-valued tables: $pq = 1$, when $p = 1$ and $q = 1$, and $pq = 3$ in the other cases; $p \vee q = 3$ when $p = 3$ and $q = 3$ and $p \vee q = 1$ in the other cases; $\sim p = 4 - p$; $X \vdash Y$ has the value 1 when and only when the value of X is greater than or equal to the value of Y. Here the theorem $p \vdash pp$ has the value 3 when $p = 2$.

We have thus shown that, with the exception of A_8, all the axioms and rules of S_2 are independent.

The axiom A_8 is independent in S_1. For proof of this it is enough to consider that the formula $X \vdash Y$ has an undesignated value if and only if it has the form $X_1 \vee X_2 \vdash Y_1 Y_2$ and no one of the following conditions is met: 1) the sign $\sim$ does not occur in $X \vdash Y$; 2) variables lacking in Y_1 or Y_2 do not occur in X; 3) the formulae $X_1 \vdash X_2$ and $X_2 \vdash X_1$ are theorems of S_1.

The independence of the rest of the axioms and rules of S_1 follows from their independence in S_2. The independence of the axioms and rules of S_3 and S_4 can be shown by the same interpretation as for the corresponding axioms and rules of S_1 and S_2.

E. A. SIDORENKO

SOME VARIANTS OF THE SYSTEMS OF LOGICAL ENTAILMENT

The essential trait of the systems S_1–S_4 is that they are unified: one system can be easily obtained from another. This facilitates the solution of many problems, especially those of independence. However, one can abstract from the unification and formulate systems of logical entailment, based on other requirements (in particular, a more simple proof of the important theorems).

A system S_1^* of strong logical entailment, equivalent to S_1, is obtained from S_1 by putting the axiom A_{11}^* and the rule R_5 for A_8 and A_{11}:

A_{11}^*. $p \vee \sim qq \vdash p$

R_5. If $X \vdash Y$, then $X \vdash Y \vee Z$, where Z does not contain any sentential variables lacking in X.

We show that S_1^* is equivalent to S_1.

T_1. $X \vdash X(p \vee \sim p)$, where p occurs in X.

1. $\sim X \vee \sim pp \vdash \sim X$ $[A_{11}^*, R_1]$
2. $\sim X \vdash \sim X \vee \sim pp$ $[R_5]$
3. $\sim\sim X \vdash \sim(\sim X \vee \sim pp)$ $[1, 2, R_2]$
4. $X \vdash X(p \vee \sim p)$ $[3; A_9, A_{10}, R_2, R_3]$

T_2. $pq \vee r \vdash (pq \vee r)(q \vee \sim q)$ $[T_1]$

T_3. $pr \vee qr \vdash (p \vee q)r$

1. $p(p \vee q) \vdash p(q \vee \sim q)$ $[T_1, A_3]$
2. $p(q \vee \sim q) \vdash p(p \vee q)$ $[A_3, R_5, R_4]$
3. $pq \vee p \dashv\vdash p \vee \sim qq$ $[1, 2, R_2, A_9, A_{10}]$
4. $pr \vee qr \vdash pr \vee qr \vee r \vee \sim pp \vee \sim qq$ $[R_5]$
5. $pr \vee qr \vdash r \vee \sim pp \vee \sim qq$ $[3, 4, R_2, R_3]$
6. $pr \vee qr \vdash r$ $[5, A_{11}^*, R_3]$
7. $pr(q \vee \sim q)(p \vee q) \vee qr(p \vee \sim p)(p \vee q) \vdash p \vee q$ $[6, R_1]$
8. $pr(q \vee \sim q) \dashv\vdash pr(q \vee \sim q)(p \vee q)$ $[A_3, R_5, R_4]$
9. $qr(p \vee \sim p) \dashv\vdash qr(p \vee \sim p)(p \vee q)$ $[A_3, R_5, R_4]$
10. $pr(q \vee \sim q) \vee qr(p \vee \sim p) \vdash pr(q \vee \sim q)(p \vee q) \vee$
    $\vee\, qr(p \vee \sim p)(p \vee q)$ $[8, 9, R_2, R_3]$

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11. $pr(q \vee \sim q) \vee qr(p \vee \sim p) \vdash p \vee q$ $[7, 10, R_3]$
12. $(pr \vee qr)(p \vee \sim p)(q \vee \sim q) \vdash pr(q \vee \sim q) \vee qr(p \vee \sim p)$ $[A_3, A_7, R_3, R_4]$
13. $pr \vee qr \vdash (pr \vee qr)(p \vee \sim p)(q \vee \sim q)$ $[T_1, R_3]$
14. $pr \vee qr \vdash p \vee q$ $[11, 12, 13, R_3]$
15. $pr \vee qr \vdash (p \vee q)r$ $[6, 14, R_4]$

Another variant of S_1, S_1^{**}, is obtained from S_1 by putting $A_9^*, A_{10}^*, A_{11}^*$ and R_2^* for A_9, A_{10}, A_{11} and R_2.

A_9^*. $\sim(p \vee q) \vdash \sim p \sim q$
A_{10}^*. $\sim p \sim q \vdash \sim(p \vee q)$

R_2^*. From $X \vdash Y$ follows $\sim Y \vdash \sim X$, where X and Y contain the same sentential variables.

In S_1^{**} the rule R_2 is proved by mathematical induction according to the number of occurrences of logical constants in the formula X. We will not carry out the proof but just indicate that the case where X has the form $X_1 \vee X_2$ and Y has the form $Y_1 \vee Y_2$ reduces to cases where X has the form X_1X_2 and $\sim X_1$ and has to be considered after them. Proof of the axioms A_3, A_{10}, A_{11} is now trivial.

Thus, we obtain a formulation of a system of strong logical entailment, in which the rule R_2 (rule of substitutability of equivalence) is not taken as basic, as the tradition holds.

In S_2 it is enough to remove the limitation of R_5 and to limit R_3 (X and Z have at least one identical variable) in order to obtain a variant of the system of weakened logical entailment (we designate it as S_2^*). The proof is evident.

If in the system S_1^{**} we eliminate the axiom A_8, remove the limitation on R_2^*, and retain R_3 with the same limitation as in S_2^*, then we obtain a variant of the system of weak logical entailment (we designate it S_2^{**}) where the rule of substitutability of equivalence is not basic. In S_2^{**} the rule R_2 is proved in the same way as in S_1^{**}, the axiom A_{12} is obtained from A_3 through R_2^*, and axiom A_8 is obtained in function of S_1^*.

A more convenient formulation of the system of maximal logical entailment is S_3^* which is obtained from S_3 by putting R_2^* and R_4^* for R_2 and R_4 and by adding the axiom A_{12}.

A_{12}^*. $p \vee p \vdash p$
R_4^*. $X_1 \vee Y_1 \vdash X_2 \vee Y_2$ follows from $X_1 \vdash X_2$ and $Y_1 \vdash Y_2$.

The rules R_2 and R_4 are simply obtained in S_3^*: R_2 is proved by mathematical induction according to the number of occurrences of logical constants in X; R_4 follows from R_2^*, R_4^*, A_9, A_{10}, A_{12}.

The system of converse logical entailment S_4^* is obtained from S_4 by putting A_{11}^{**} and R_5^* for the axiomatic schema A_3 and the axioms A_8, A_{11}.

A_{11}^*. $p \vdash p(q \vee \sim q)$

R_5^*. From $X \vdash Z$ follows $XY \vdash Z$, where Y does not contain variables lacking in Z.

The axioms A_3 and A_{11} are obtained trivially in S_4^* and, with some obvious changes, the proof of A_8 in S_1^* can be taken for A_8.

The system of converse entailment, where the rule of substitutability of equivalence is not taken as basic, is obtained from S_3^* by adding A_{12}. obviating the necessity for A_3. The proof is evident.

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COMPLETENESS OF THE SYSTEMS OF LOGICAL ENTAILMENT

For the proof of the completeness of S_1 Zinov'ev formulated a series of lemmas. Below we will offer proofs of these lemmas.

Proof of L_1.

1. $X^1 \cdot \ldots \cdot X^n \dashv\vdash Y$, where Y differs from $X^1 \cdot \ldots \cdot X^n$ by any arrangement of brackets corresponding to a definition of a sentential formula, or by the order of writing $X^1, \ldots, X^n$ [A_4–A_6, R_1, R_2, R_4]

2. $X^1 \vee \ldots \vee X^n \dashv\vdash \sim (\sim X^1 \cdot \ldots \cdot \sim X^n)$ [A_9, A_{10}, A_1, A_2, R_1, R_4]
3. $X^1 \vee \ldots \vee X^n \dashv\vdash Y$,

where Y differs from $X^1 \vee \ldots \vee X^n$ by arrangement of brackets or order of writing $X^1, \ldots, X^n$. [1, 2, R_4, R_2].

4. $X \cdot X \dashv\vdash X$ [A_3, A_1, A_2, R_1, R_2, R_3]
5. $X \vee X \dashv\vdash X$ [1, A_9, A_{10}, R_4, R_2, A_1, A_2]
6. $(p \vee q)\, r \dashv\vdash pr \vee qr$ [A_8, A_7, R_1, A_3, R_3]
7. $X \dashv\vdash X(p \vee \sim p)$,

where p occurs in X.

Theorem 7 is proved by induction according to the number of occurrences of logical constants in X. Let X be p. Then

$$p \dashv\vdash p(p \vee \sim p) \; [A_1, A_2, R_2, A_{11}, 6, 7, R_3, A_3].$$

If X is $\sim p$, we obtain

$$\sim p \dashv\vdash \sim p(p \vee \sim p).$$

Let X be $Y \vee Z$. If p occurs in Y, then according to the inductive assumption

$$Y \dashv\vdash Y(p \vee \sim p).$$

Consequently, we obtain

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a) $Y \vee Z \dashv\vdash Y(p \vee \sim p) \vee Z$ $[R_4, R_2]$
b) $Y \vee Z \dashv\vdash Yp \vee (Y \sim p \vee Z)$ $[6, R_4, R_{11}, R_2, 3]$
c) $Y \vee Z \dashv\vdash (Y(p \vee \sim p) \vee Z)(p \vee \sim p)$ $[R_4, R_2, 6, R_1]$
d) $Y \vee Z \dashv\vdash (Yp \vee Y \sim p \vee Z)(p \vee \sim p)$ $[R_1, A_{11}, 3, R_4, R_2]$
e) $Y \vee Z \dashv\vdash (Y \vee Z)(p \vee \sim p)$ $[R_1, A_7, 6, R_4, R_2]$.

We obtain a similar result if p occurs in Z. Further, let X be YZ. Let p occur in Y. According to the inductive assumption

$$Y \dashv\vdash Y(p \vee \sim p).$$

In such a case we obtain

a) $YZ \dashv\vdash Y(p \vee \sim p)Z$ $[R_1, A_1, A_2, 1, R_4, R_2]$
b) $YZ \dashv\vdash (YZ)(p \vee \sim p)$ $[R_1, A_4, R_4, R_2, 1]$.

We obtain a similar result if p occurs in Z. The case where X is $\sim Y$ is reduced to that examined earlier.

Proof of L_2.

1. $\sim (X(p \vee \sim p)) \dashv\vdash \sim X$ $[R_4, L_1^7, R_2, A_1, A_2]$
2. $\sim X \vee p \cdot \sim p \dashv\vdash \sim X$ $[R_4, R_2, R_1, A_9, A_{10}, L_1^2]$
3. $X \vee p \cdot \sim p \dashv\vdash X$ $[R_1, R_4, R_2, A_1, A_2]$
4. $pq \vee r \dashv\vdash \sim\sim (pq \vee r)$ $[R_1, A_1, A_2]$
5. $\sim (pq \vee r) \dashv\vdash (\sim p \vee \sim q) \cdot \sim r$ $[A_9, A_{10}, L_1^2, R_1]$
6. $\sim (pq \vee r) \dashv\vdash \sim p \cdot \sim r \vee \sim q \cdot \sim r$ $[R_4, R_2, 5, L_1^6]$
7. $pq \vee r \dashv\vdash \sim (\sim p \cdot \sim r \vee \sim q \cdot \sim r)$ $[R_4, R_2, 4, 6]$
8. $pq \vee r \dashv\vdash (\sim\sim p \vee \sim\sim r) \cdot (\sim\sim q \vee \sim\sim r)$
$[R_4, R_2, 7, R_1, A_9, A_{10}, L_1^2]$
9. $pq \vee r \dashv\vdash (p \vee r)(q \vee r)$ $[R_4, R_2, R_1, A_1, A_2]$
10. $X \vdash X \vee Y$,

where Y contains only those variables which occur in X.
$[R_1, A_1, A_2, R_2, R_5]$
11. $X \dashv\vdash X \vee \sim pp^Y$, where no variables lacking in X occur in $\sim pp^Y$.
$[10, R_1, 9, A_3, 3]$.

Proof of L_3.

1. For every sentential formula X one can find a sentential formula Y, occurring in a normal form such that $X \dashv\vdash Y$. This assertion is proved by mathematical induction according to the number of logical constants

encountered in X. If X is p, then

$$p \dashv\vdash p \vee {\sim} pp \;\; [R_1, L_2^3].$$

Similarly

$$\sim p \dashv\vdash {\sim} p \vee {\sim} pp.$$

Let X be $X^1 \vee X^2$. According to the inductive assumption

$$X^1 \dashv\vdash Y^1$$
$$X^2 \dashv\vdash Y^2,$$

where Y^1 and Y^2 occur in normal form. In such a case

$$X^1 \vee X^2 \dashv\vdash Y^1 \vee Y^2 \qquad [R_4, R_2]$$
$$Y^1 \vee Y^2 \dashv\vdash (Y^1 \vee Y^2)(q^1 \vee {\sim} q^1) \;\; [R_1, L_1^7],$$

where q^1 is a variable lacking in Y^1 or in Y^2.

$$Y^1 \vee Y^2 \dashv\vdash Y^1(q^1 \vee {\sim} q^1) \vee Y^2(q^1 \vee {\sim} q^2) \quad [R_4, R_2, L_1^6].$$

Let Y^1 be $Z^1 \vee \ldots \vee Z^n$ and Y^2 be $Z_1 \vee \ldots \vee Z_m$.
According to L_1^6 we have

$$Y^1 \vee Y^2 \dashv\vdash Z^1 q^1 \vee \ldots \vee Z^n q^1 \vee Z^1 {\sim} q^1 \vee \ldots \vee Z^n$$
$${\sim} q^1 \vee Z_1 q^1 \vee \ldots \vee Z_m {\sim} q^1.$$

Similarly, for the other variables q^i, lacking in Y^1 or Y^2, we obtain

$$Y^1 \vee Y^2 \dashv\vdash Z^1 q^1 \ldots q^k \vee \ldots \vee Z^n q^1 \ldots q^k \vee \ldots \vee Z_1$$
$${\sim} q^1 \ldots {\sim} q^k \vee \ldots \vee Z_m {\sim} q^1 \ldots {\sim} q^k.$$

Using L_1^4 and L_1^5, R_4 and R_2, we obtain

$$Y^1 \vee Y^2 \dashv\vdash Y,$$

where Y occurs in normal form and

$$X^1 \vee X^2 \dashv\vdash Y.$$

Let X be $X^1 X^2$. According to the inductive assumption

$$X^1 \dashv\vdash Y^1$$
$$X^2 \dashv\vdash Y^2,$$

where Y^1 and Y^2 occur in normal form. Let Y^1 be $Z^1 \vee \ldots \vee Z^n$ and Y^2 be $Z_1 \vee \ldots \vee Z_m$.

We have

$$Y^1 Y^2 \dashv\vdash Z^1 Z_1 \vee \dots \vee Z^1 Z_m \vee \dots \vee Z^n Z_1 \vee \dots \vee Z^n Z_m$$
$$[R_4, R_2, L_1^6]$$

Using L_1^1, L_1^4 and L_1^5, we obtain

$$Y^1 Y^2 \dashv\vdash Y,$$

where Y occurs in normal form. Since

$$X^1 X^2 \dashv\vdash Y^1 Y^2$$

we have

$$X^1 X^2 \dashv\vdash Y.$$

The case where X is $\sim X^1$ is reduced to that considered above.

2. Let $p^1, \dots, p^k$ be all variables occurring in X and lacking in Y.

$$X \vdash Y(p^1 \vee \sim p^1) \dots (p^k \vee \sim p^k) \ [L_1^7, R_3, R_2].$$

According to L_3^1, one can find for X a formula X^* in normal form such that

$$X \dashv\vdash X^*$$

Similarly, for $Y(p^1 \vee \sim p^1) \dots (p^k \vee \sim p^k)$ one can find a formula in normal form such that

$$Y(p^1 \vee \sim p^1) \dots (p^k \vee \sim p^k) \dashv\vdash Y^*$$

and by definition $X^* \vdash Y^*$ is the normal form of $X \vdash Y$.

The lemmas L_4 and L_5 are evident. Lemma L_6 is proved by Zinov'ev.

The proof of the completeness of S_1 holds for S_2 since in the proof of all the theorems considered above, L_1–L_6, the limitation of R_2 is satisfied. The proof of L_3^2 is only slightly modified by the following addition. Let $q^1, \dots, q^l$ be all variables occurring in Y and lacking in X. In such a case

$$X(q^1 \vee \sim q^1) \dots (q^l \vee \sim q^l) \vdash Y(p^1 \vee \sim p^1) \dots (p^k \vee \sim p^k).$$

And for $X(q^1 \vee \sim q^1) \dots (q^1 \vee \sim q^1)$ there is a formula X^* in normal form such that

$$X(q^1 \vee \sim q^1) \dots (q^l \vee \sim q^l) \dashv\vdash X^*.$$

To prove the completeness of S_3 it is enough to prove L_1^6 in such a way that the limitation on A_3 be satisfied. It takes the form

1. $(p \vee q)\, r \vdash (p \vee q)\, r$ $[R_2, R_1, A_1, A_2]$
2. $(p \vee q)\, r \vdash (pr \vee q)$ $[A_7]$
3. $(p \vee q)\, r \vdash (p \vee q)\, r (pr \vee q)$ $[R_3, 1, 2]$
4. $(p \vee q)\, r (pr \vee q) \vdash (pr \vee q)\, r$ $[R_6]$
5. $(pr \vee q)\, r \vdash pr \vee qr$ $[R_1, A_7, A_4]$
6. $(p \vee q)\, r \vdash pr \vee qr$ $[R_2, 3, 4, 5]$
7. $(pr \vee qr) \vdash (p \vee q)\, r$ $[A_8]$

The limitation on A_3 is satisfied in the proof of the other lemmas.

In S_4 there are the following changes in the proof of the theorems: L_1^6 is proved as in S_3. The proof of L_3^2 disappears since the theorem on completeness is proved as follows. Let $X \vdash Y$ be a tautology. Then, by virtue of L_3^1, $X^1 \vdash Y^1$, where X^1 and Y^1 are, respectively, the normal forms of X and Y, is also a tautology. Putting expressions like $q_i(p_i \vee \sim p_i)$, where p_i is a variable occurring in Y but not in X, for the sentential variable q_i which occurs in X and therefore in Y, also issues in a tautology. Application of L_1^6, R_1, R_4 to the resultant expression provides us with a type of disjunction-conjunction of all variables (or their negations) occurring in X, plus p_i. Thus, in X^1 are introduced all variables occurring in Y but lacking in X. Recurrence of sentential variables in Y is eliminated by R_4, L_1^4. We here obtain the formula of entailment $X^* \vdash Y^*$ which conserves the property of being a tautology and which also has the property that the same variables occur in the antecedent and consequent. Further proof of the theorem of completeness is the same as in S_1.

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COMPLETENESS OF SYSTEMS OF DEGENERATE ENTAILMENT AND QUASI-ENTAILMENT

Below we will prove the completeness of the systems S_5 and S_6, proposed by Zinov'ev.

T_1. If X is a tautology, then $\vdash X$ is provable in S_5.

Before proving T_1 it is necessary to prove two lemmas:

L_1. For every sentential formula X one can find a sentential formula Y in normal form such that $X \vdash Y$ and $Y \vdash X$ are provable in S_5.

L_2. If Y is a tautology in normal form, then $\vdash Y$ is provable in S_5.

The lemma L_1 is proved in Fedina's article.

Proof of L_2. Let only one variable, p, occur in Y. In such a case Y is $p \vee \sim p$. But $\vdash (p \vee \sim p)$ is provable in function of $A_{12}, A_9, A_{10}, R_1, R_4$. Let it be the case that n variables $p^1, \ldots, p^n$ $(n \geqslant 2)$ occur in X. In such a case Y has the form $Y^1 \vee \ldots \vee Y^{2^n}$, where $Y^1, \ldots, Y^{2^n}$ are all possible conjunctions formed from $p^1, \ldots, p^n$ and their negations.

1. $((p^1 \cdot \ldots \cdot p^n) \vee \sim (p^1 \cdot \ldots \cdot p^n))$ $\quad [\vdash (p \vee \sim p), R_1]$
2. $((p^1 \cdot \ldots \cdot p^n) \vee (\sim p^1 \vee \ldots \vee \sim p^n))$ $\quad [A_9, A_{10}, R_4]$
3. $(p^1 \cdot \ldots \cdot p^n) \vee (\sim p^1 \vee \ldots \vee \sim p^n) \vdash ((p^1 \cdot \ldots \cdot p^n) \vee (\sim p^1 \vee \ldots \vee \sim p^n))$ $(p^n \vee \sim p^n)$ $\quad [A_{11}, R_1]$
4. $\vdash ((p^1 \cdot \ldots \cdot p^n) \vee (\sim p^1 \vee \ldots \vee \sim p^n))(p^n \vee \sim p^n)$ $\quad [2, 3, R_5]$
5. $\vdash ((p^1 \cdot \ldots \cdot p^n)(p^n \vee \sim p^n) \vee (\sim p^1 \vee \ldots \vee \sim p^n))(p^n \vee \sim p^n)$ $\quad [4, R_4, (p \vee q)\, r \dashv\vdash pr \vee qr]$
6. $(p^1 \cdot \ldots \cdot p^n)(p^n \vee \sim p^n) \dashv\vdash (p^1 \cdot \ldots \cdot p^n)$ $\quad [X(p \vee \sim p) \dashv\vdash X, R_1]$
7. $(\sim p^1 \vee \ldots \vee \sim p^n)(p^n \vee \sim p^n) \dashv\vdash \sim p^1 (p^n \vee \sim p^n) \vee \ldots \vee \sim p^n (p^n \vee \sim p^n)$ $\quad$ [similar to 5]
8. $\vdash (p^1 \cdot \ldots \cdot p^n) \vee (\sim p^1 (p^n \vee \sim p^n) \vee \sim p^2 (p^n \vee \sim p^n) \vee \ldots \vee \sim p^n))$ $\quad [5, 6, 7, R_4]$

and so on for the remaining $n-1$ variables. The result is that the formula

$$\vdash ((p^1 \cdot \ldots \cdot p^n) \vee \sim p^1 (p^n \vee \sim p^n) \cdot \ldots \cdot (p^2 \vee \sim p^2) \vee \ldots \vee \sim p^n (p^{n-1} \vee \vee \sim p^{n-1}) \cdot \ldots \cdot (p^1 \vee \sim p^1)$$

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is provable. Whence it follows, according to A_5–A_8, that the formula $Y^1 \vee \ldots \vee Y^{2^n}$ is provable.

If Y is a normal form of a formula X, then $Y \vdash X$ is provable (by definition). In function of L_2 and the rule "If $Y \vdash X$ and $\vdash Y$, then $\vdash X$" we obtain $\vdash X$.

T_2. If $X \supset Y$ is a tautology, then $X \rightarrow Y$ is provable in S_6.

Proof of T_2. Let $X \supset Y$ be a tautology. Three cases of the occurrence of variables in X and Y are possible: 1) the set of variables occurring in Y is identical with that occurring in X; 2) only those variables which occur in Y occur in X; 3) Y contains at least one variable which does not occur in X.

If 1) and 2) are the case then $X \rightarrowtail Y$ is provable in S_6: if $X \supset Y$ is a tautology of the type 1) and 2), then $X \vdash Y$ is provable in S_1 according to the theorem on the completeness of S_1. But according to the definition of the proved formula of quasi-entailment we have: if $X \sim (p \sim p) \vdash Y$ and $\vdash \sim (\sim pp)$ are provable, then $X \rightarrowtail Y$ is provable. The formula $\vdash \sim (\sim pp)$ is clearly provable. The formula $X \sim (\sim pp) \vdash X$ is provable according to A_3 and R_1, $X \sim (\sim pp) \vdash Y$ according to R_2 and the formulae proved above. Thus, $X \rightarrowtail Y$ is provable.

Let us examine case 3). Let $p^1, \ldots, p^n$ $(n \geqslant 1)$ be all the variables which occur in Y and not in X. In such a case, the formula $X \cdot ((p^1 \cdot \ldots \cdot p^n) \vee \sim (p^1 \cdot \ldots \cdot p^n)) \supset Y$ will be a tautology in exactly the same way, and the formula $X \cdot ((p^1 \cdot \ldots \cdot p^n) \vee \sim (p^1 \cdot \ldots \cdot p^n)) \vdash Y$ will be provable in S_1 in function of the completeness theorem. But the formula $\vdash ((p^1 \cdot \ldots \cdot p^n) \vee \sim (p^1 \cdot \ldots \cdot p^n))$ is provable in S_6. Consequently, $X \rightarrowtail Y$ is provable according to the definition of a provable formula of quasi-logical entailment.

The system of quasi-entailment S_6^1, equivalent to S_6, is obtained by dropping the limitations on R_2.

T_3. If $X \supset Y$ is a tautology (provable in classical sentential calculus), then $X \rightarrowtail Y$ is provable in S_6^1.

The lemma L_1 for S_6^1 is proved analogously to S_6.

L_3. The formula $p \dashv\vdash p(q \vee \sim q)$ is provable in S_6^1.

1. $p(q \vee \sim q) \vdash p$ $[A_3, R_1]$
2. $p \vdash rq \vee p$ $[A_{12}, R_1, R_2, p \vee q \vdash q \vee p]$
3. $rq \vee p \vdash (rq \vee p)(q \vee \sim q)$ $[A_{11}, R_1]$
4. $(rq \vee p)(q \vee \sim q) \vdash (q \vee \sim q)$ $[A_3, R_1]$

5. $(rq \vee p) \vdash q \vee \sim q$ $[3, 4, R_2]$
6. $p \vdash q \vee \sim q$ $[2, 5, R_2]$
7. $p \vdash p$ $[A_1, A_2, R_2]$
8. $p \vdash p(q \vee \sim q)$ $[6, 7, R_3]$

L_4. Let $p^1, \ldots, p^n$ be all variables occurring in Y and not in X. If $X(p^1 \vee \sim p^1) \cdot \ldots \cdot (p^n \vee \sim p^n) \supset Y$ is a tautology, then $X(p^1 \vee \sim p^1) \cdot \ldots \cdot (p^n \vee \sim p^n) \vdash Y$ is provable in S_1, which means in S_6^1, too. But $X \vdash X(p^1 \vee \sim p^1) \cdot \ldots \cdot (p^n \vee \sim p^n)$ is provable in S_6^1 in function of L_3. Consequently, in function of R_2, $X \rightarrowtail Y$ is provable.

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